LOCALITY AT THE BOUNDARY IMPLIES GAP IN THE BULK FOR 2D PEPS

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Based on joint work with Angelo Lucia and David Perez-Garcia

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RESULT

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Motivation:

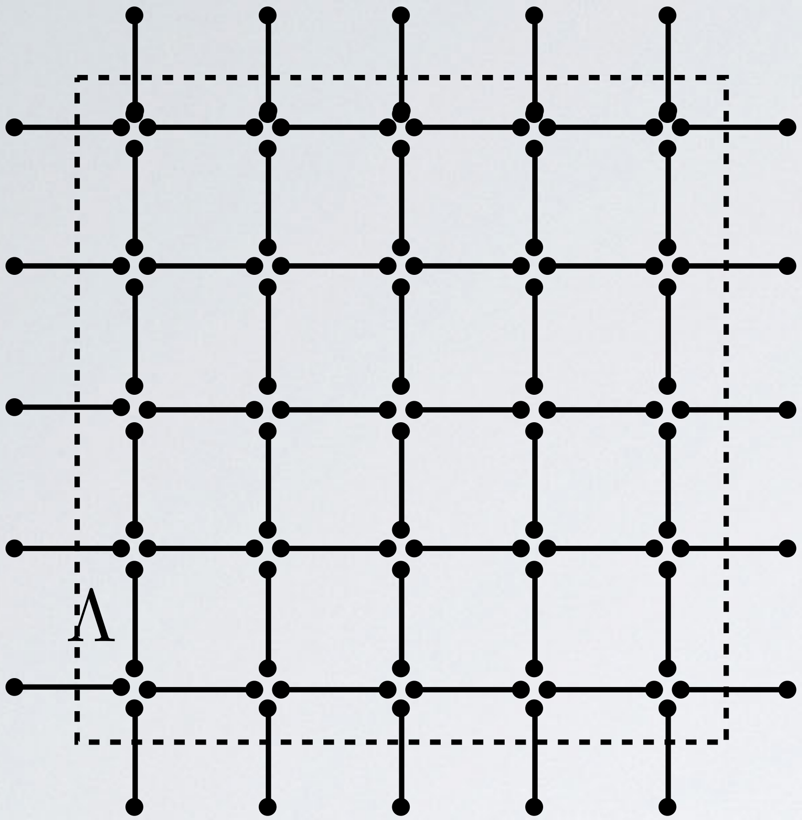
- Classification of phases in 2D
- Rigorous bulk boundary correspondence
- Showing that specific models are gapped:
  Ex. The 2D AKLT on hexagonal lattice

OUTLINE

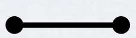
If the boundary state of an (injective) PEPS is Gibbs (or sufficiently local), then its parent Hamiltonian is gapped.

1) PEPS basics

PEPS BASICS



- Λ is a finite subset of an (infinite) graph (G, E); Typically a lattice.
- Associate to each edge $e \in E_{\bar{\Lambda}}$ a maximally entangled state



$$|\omega_e\rangle = D^{-1} \sum_{j=1}^{D} |jj\rangle$$

D is the bond dimension

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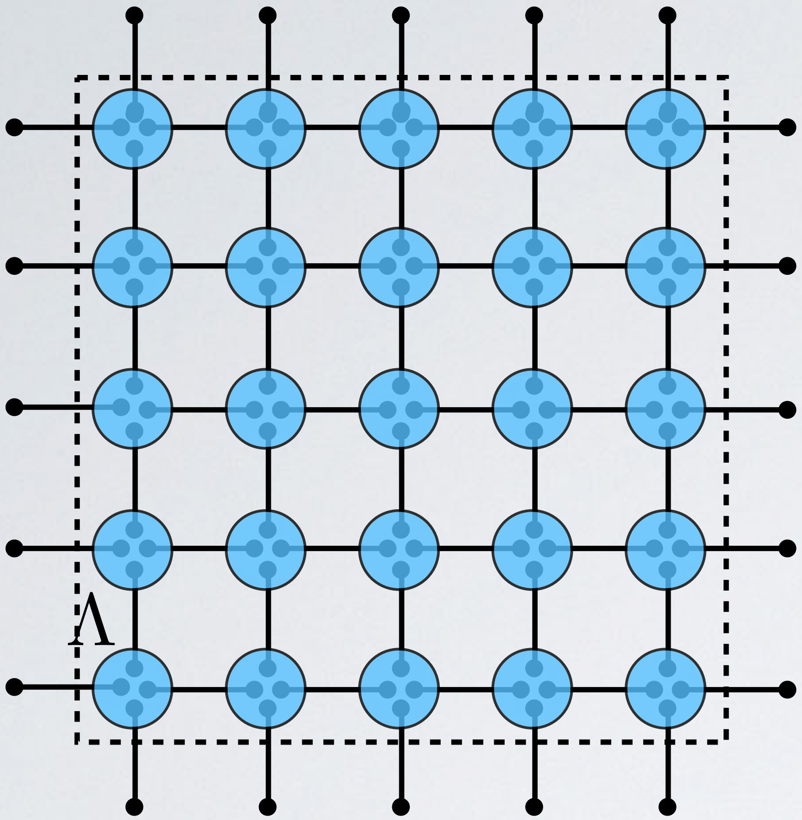

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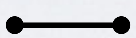

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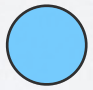
- Associate to each vertex $v \in \Lambda$ a linear map



$$T_v : \mathcal{H}_D^{\otimes deg(v)} \rightarrow \mathcal{H}_d$$

d is the dimension of the local physical space.

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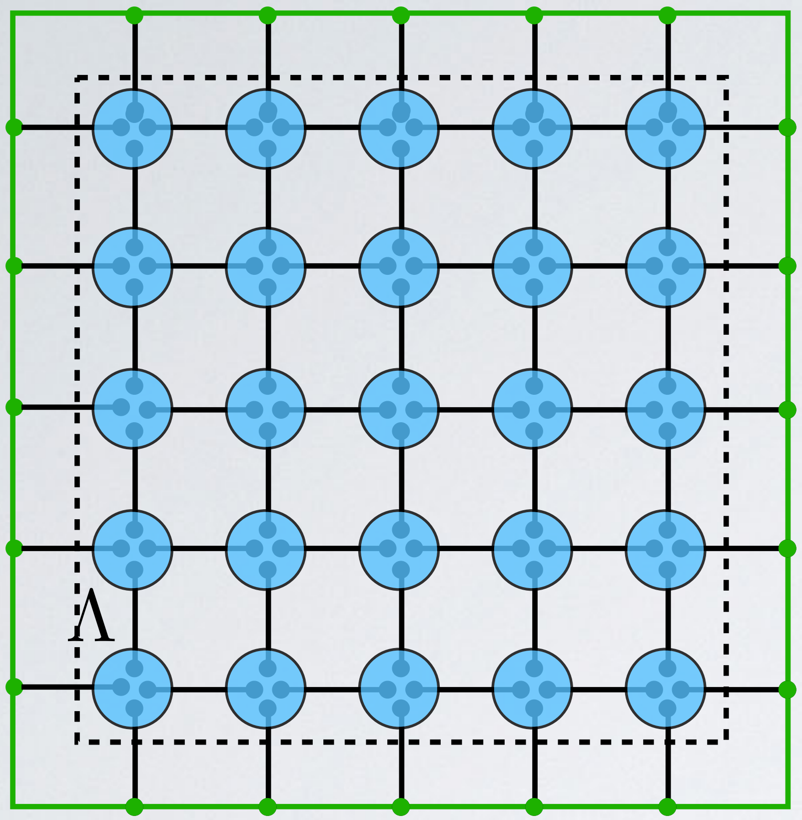

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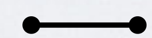

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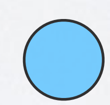

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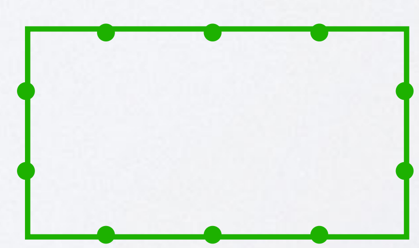

$$|\chi_{\partial\Lambda}\rangle$$

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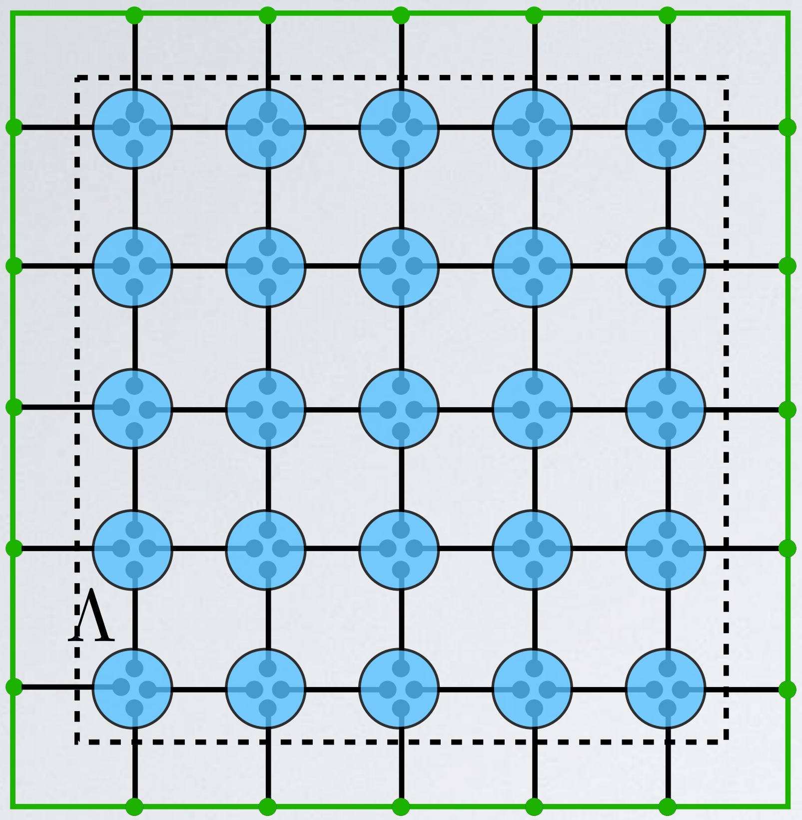

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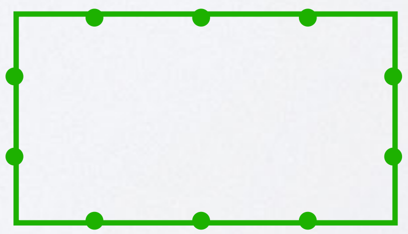
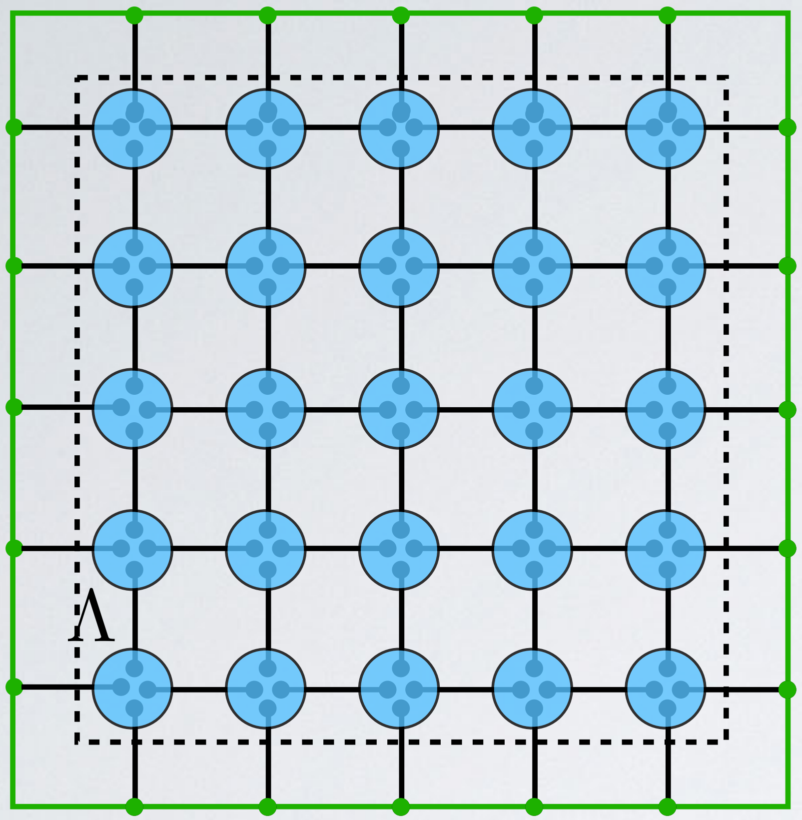


$$|\chi_{\partial\Lambda}\rangle$$

The PEPS on Λ with boundary condition $\chi_{\partial\Lambda}$ is given by

$$|\Psi_{\Lambda,\chi}\rangle = \langle \chi_{\partial\Lambda}| \bigotimes_{v\in\Lambda} T_v \bigotimes_{e\in E_{\bar{\Lambda}}} |\omega_e\rangle$$

WHY PEPS?



- Efficient numerical algorithms
- Efficient local description of topological order
- Believed to accurately characterise ground states of gapped many body Hamiltonians
- If you're not yet convinced, go have a beer with Frank or Norbert…

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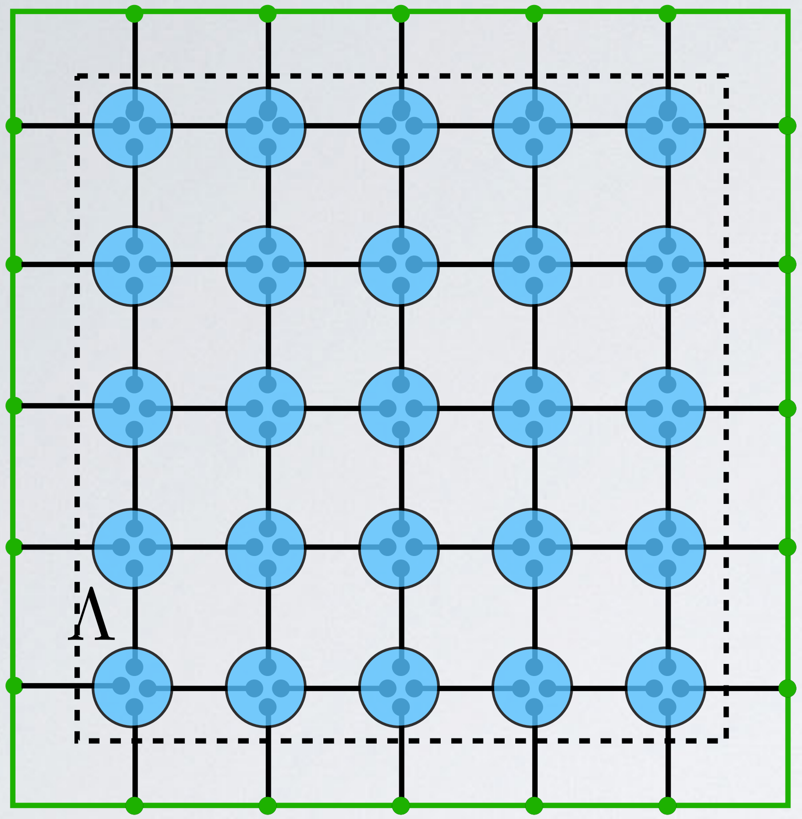
INJECTIVITY



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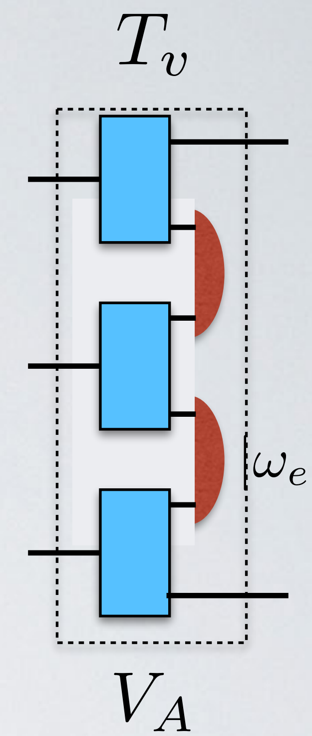
- For convenience, we define a map

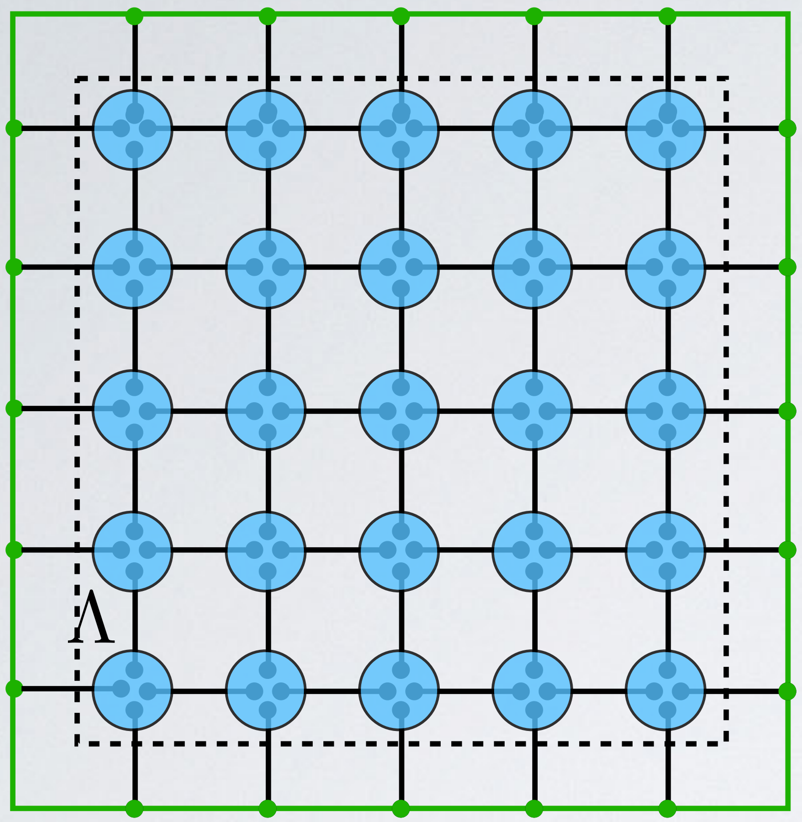
$$V_\Lambda = \bigotimes_{v\in\Lambda} T_v \bigotimes_{e\in E_{\bar{\Lambda}}} |\omega_e\rangle$$

V_Λ is a mapping from the boundary Hilbert space to the bulk Hilbert space



- The PEPS is injective on Λ if the map $V_\Lambda : \mathcal{H}_{\partial\Lambda} \to \mathcal{H}_\Lambda$ is injective.
- The PEPS is injective, if if is injective on all sufficiently large squares.

PARENT HAMILTONIAN



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- A ‘parent’ Hamiltonian can be constructed:

$$H_\Lambda = \sum_{e\in E_\Lambda} h_e$$

with local terms $h_e = T^{-1}_{nbh(e)}(1 - |\omega_e\rangle\langle\omega_e|)T^{-1}_{nbh(e)}$

and $T_{nbh(e)} = \bigotimes_{v\in nbhd(e)} T_v$

- The parent Hamiltonian is frustration free:

$h_e|\Psi_{\Lambda,\chi}\rangle = 0$ for all $e \in E_\Lambda$

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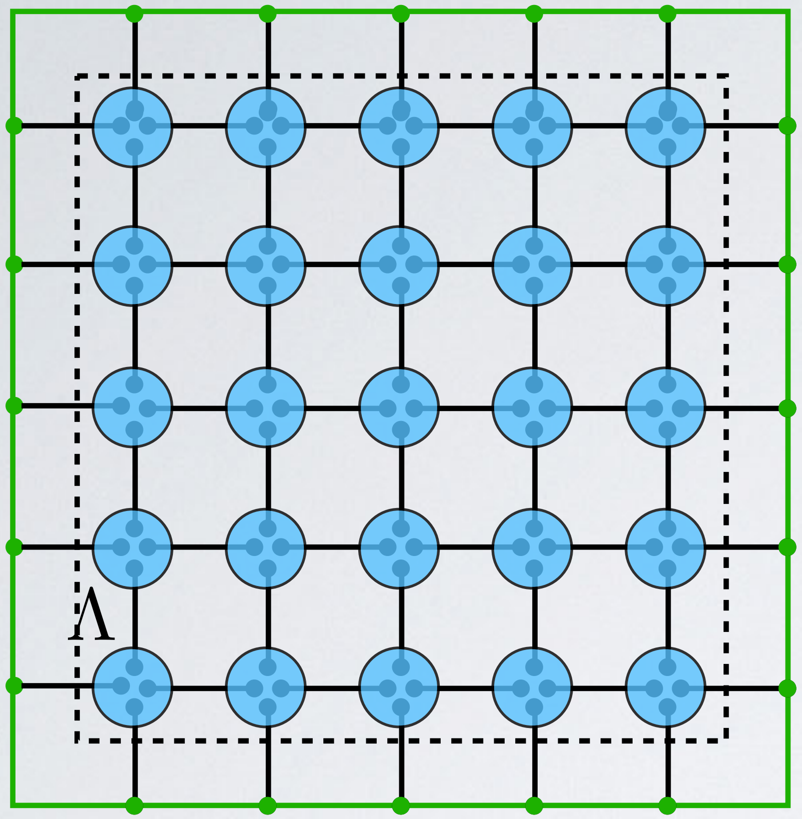

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We want to infer properties of the parent Hamiltonian from properties of the state!

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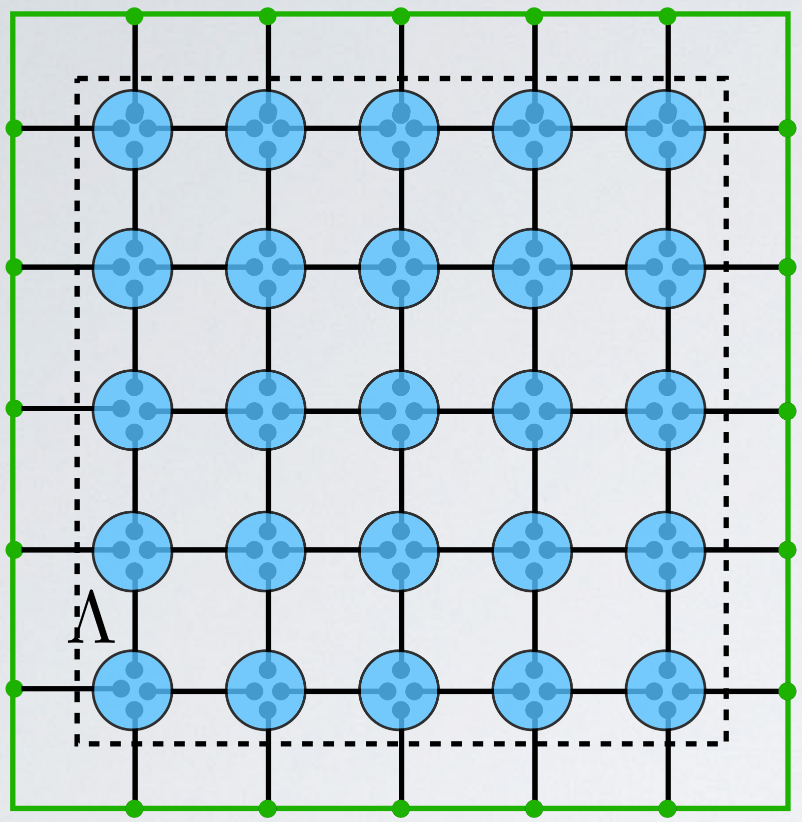

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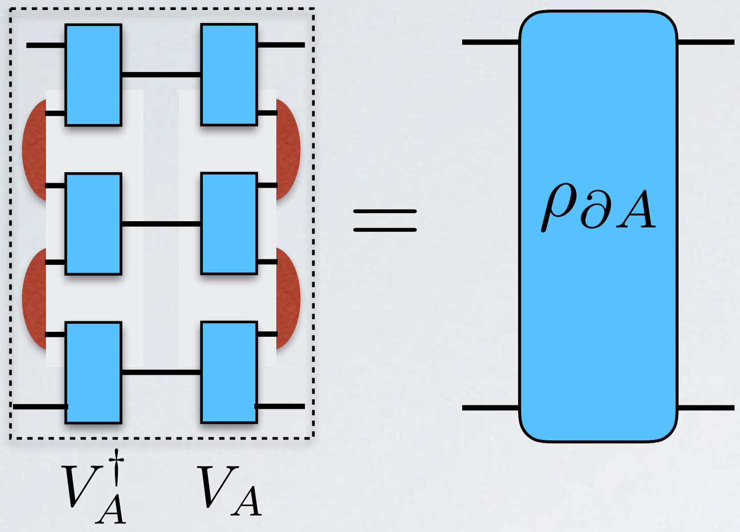
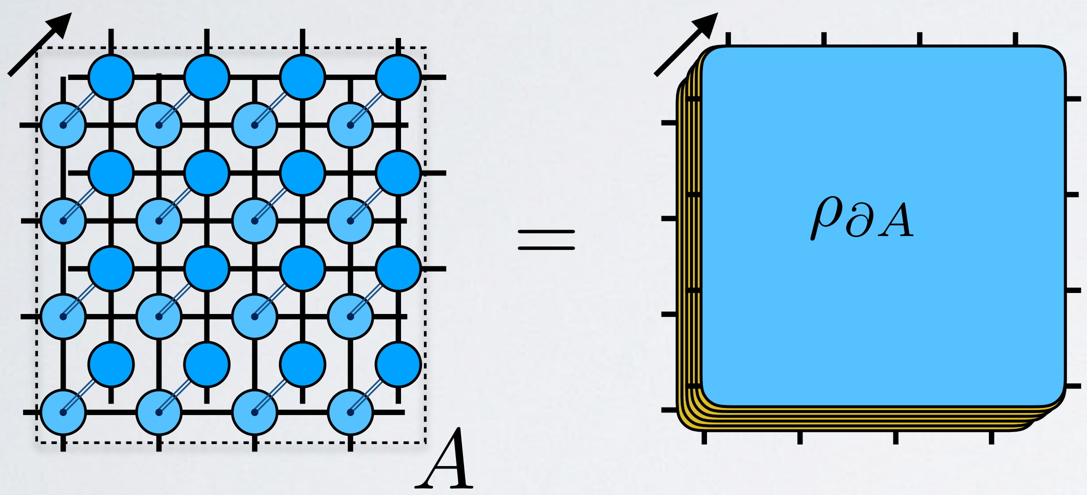
➡ But there exist critical injective PEPS!

Perez-Garcia, Verstraete, Wolf, Cirac, ‘07

Theorem: the parent Hamiltonian of an injective MPS is gapped. Fannes, Nachtergaele, Werner, ‘92

BOUNDARY STATES





- The boundary state is obtained by contracting the physical indices and keeping the virtual ones open

$$\rho_{\partial A} = V_A^\dagger V_A$$

- Properties of the boundary state:
  - $\rho_{\partial A} \geq 0$
  - $\ker \rho_{\partial A} = \ker V_A$
- If the PEPS is injective, then
  - $\rho_{\partial A} > 0$

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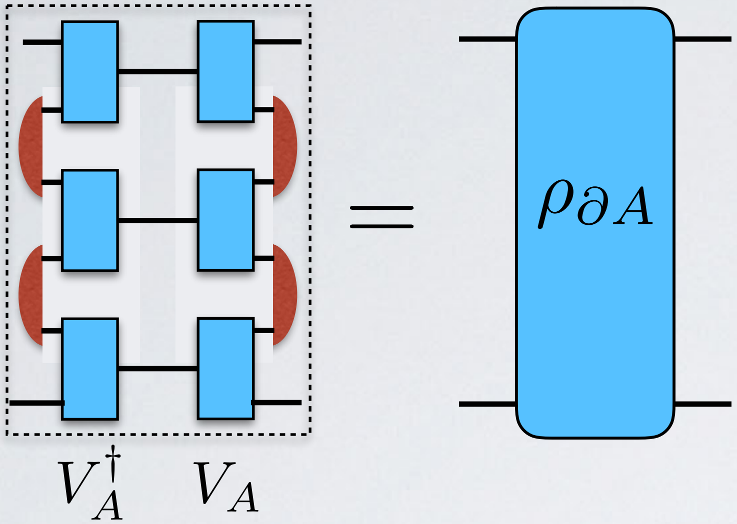

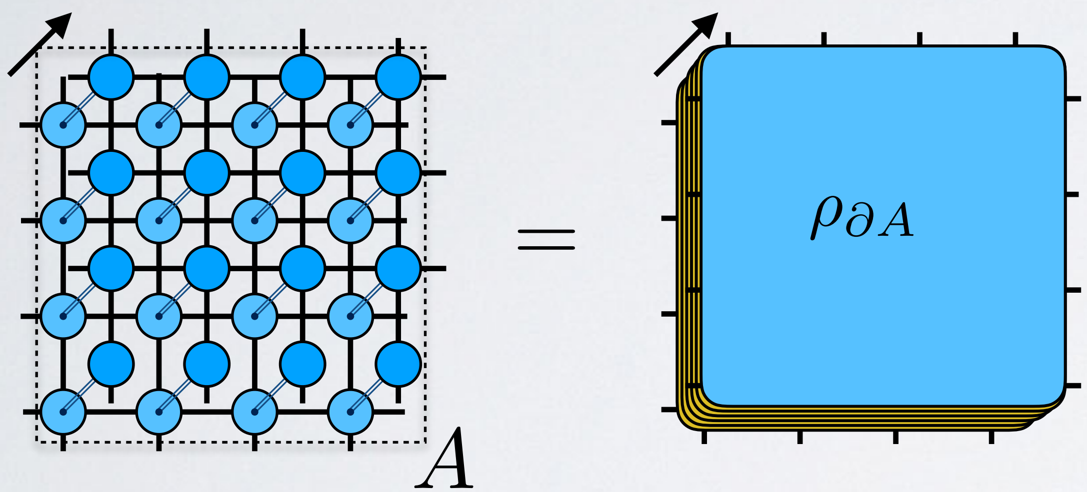

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$$\rho_{\partial A} = V^\dagger V_{A}$$

Naturally related to the entanglement spectrum!

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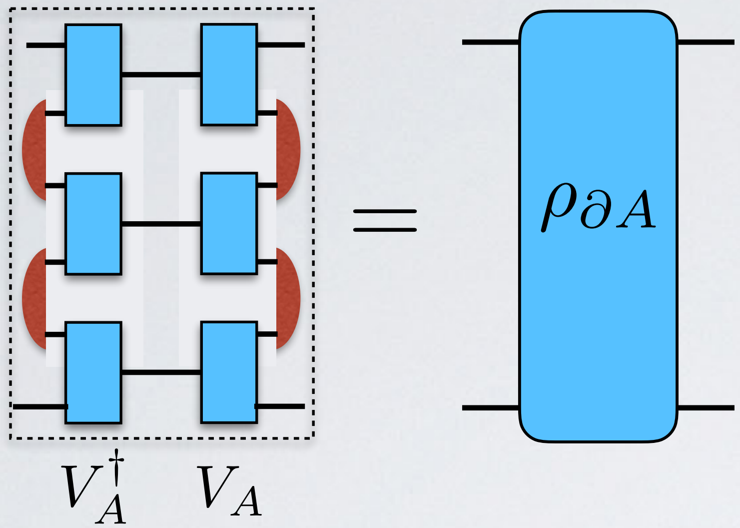

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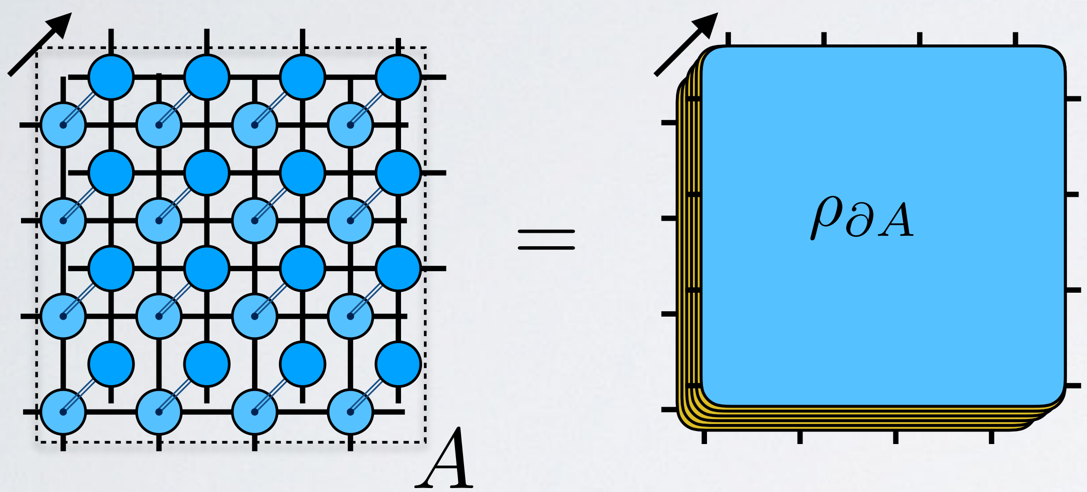

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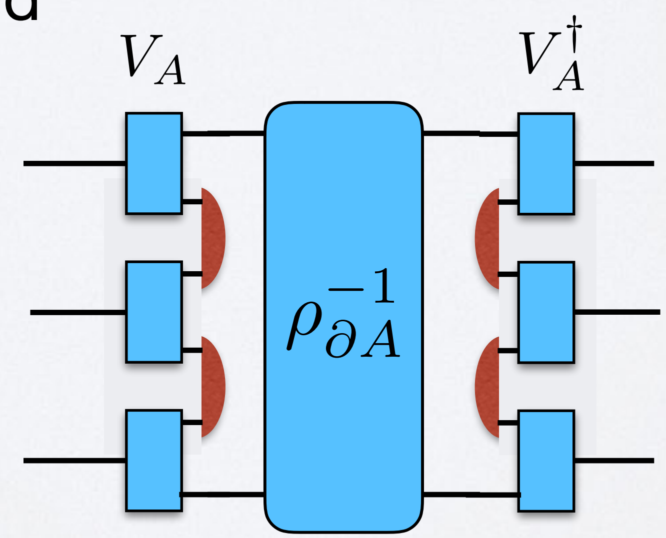

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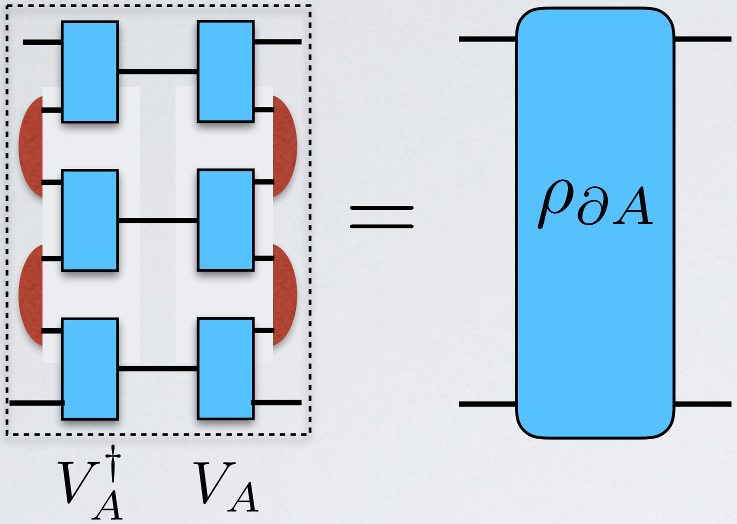

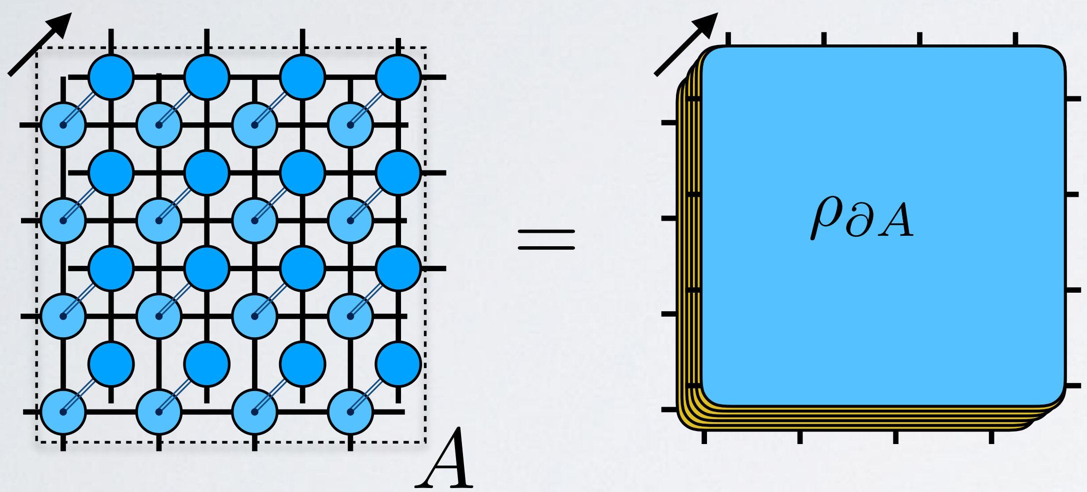

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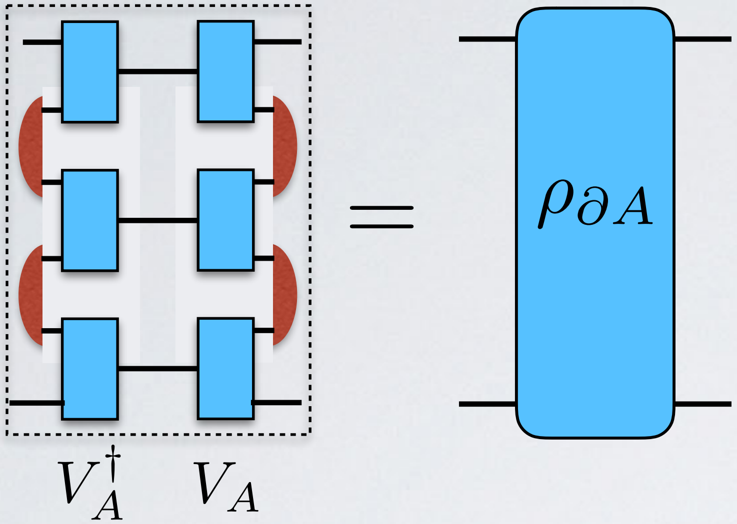

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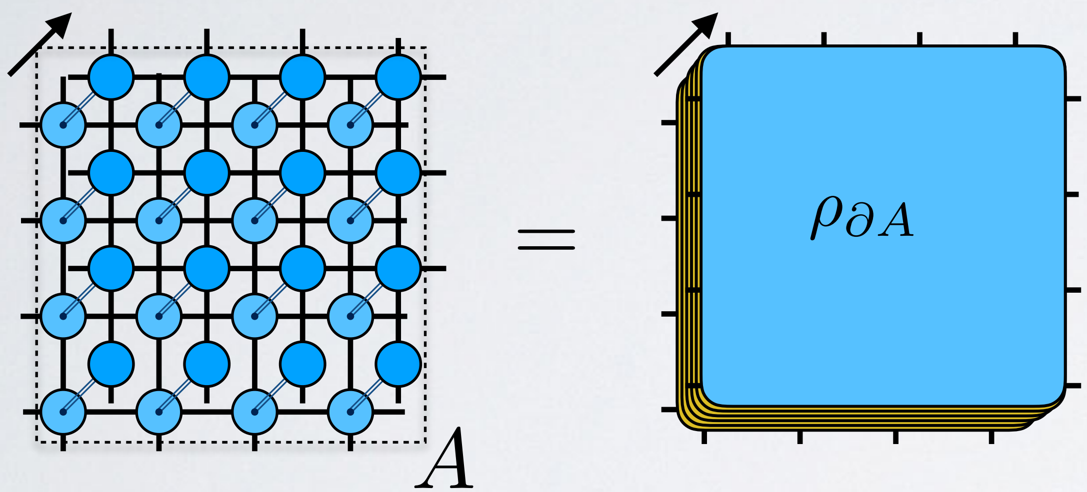

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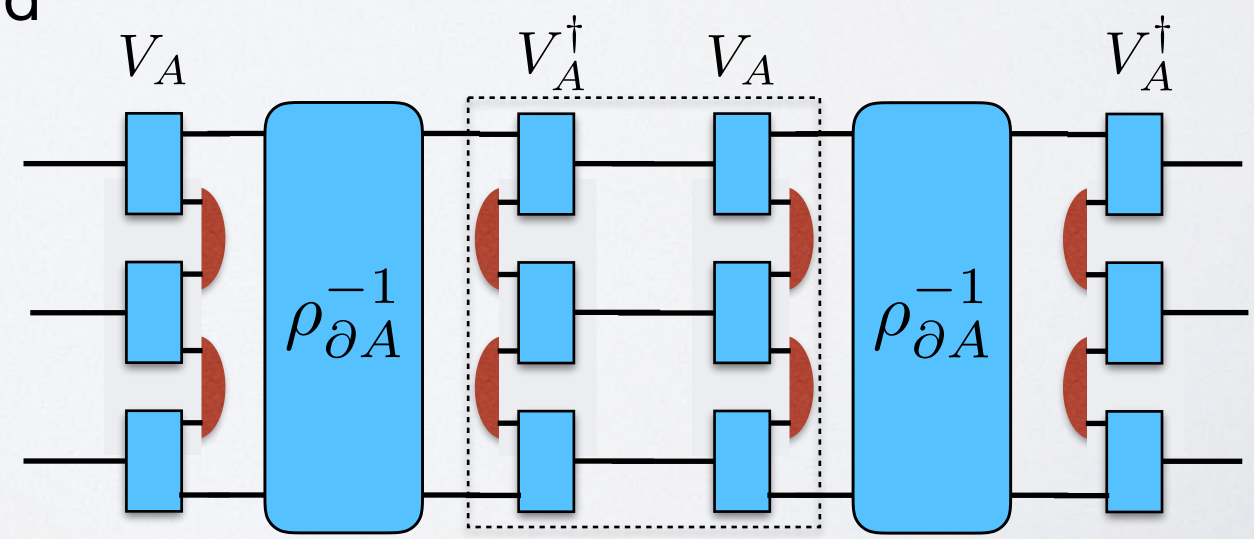

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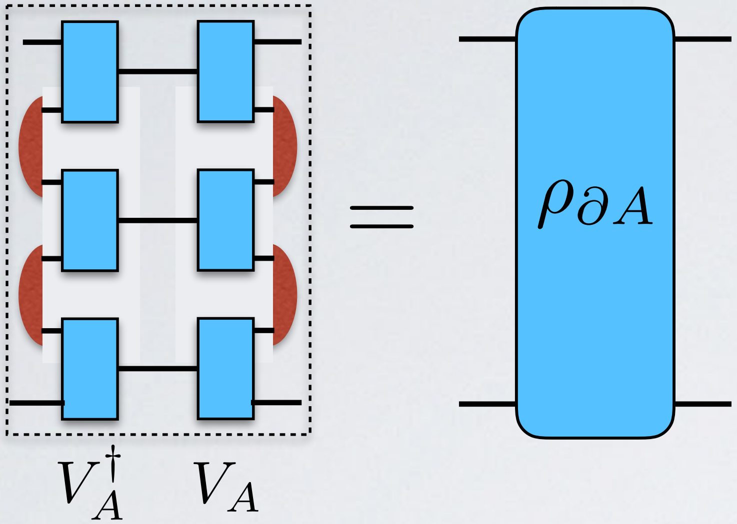

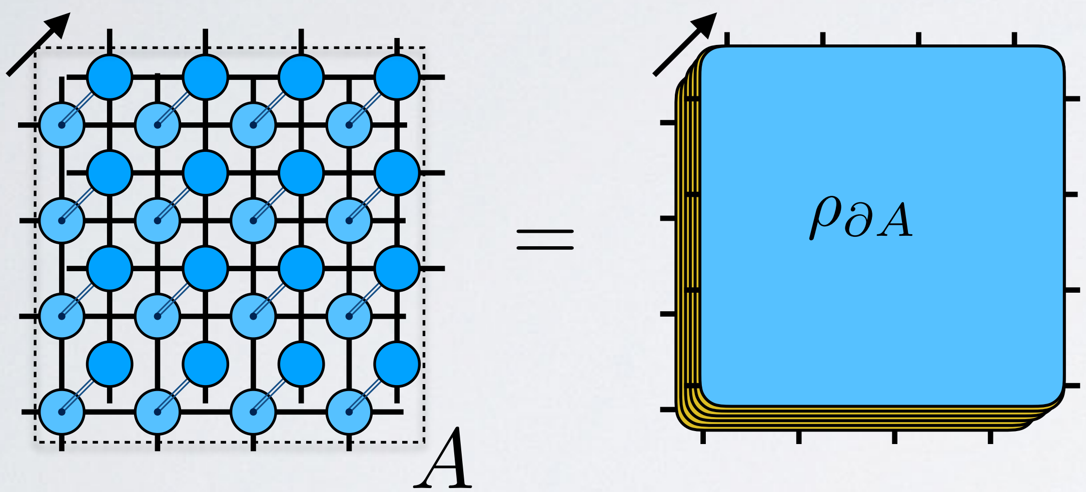

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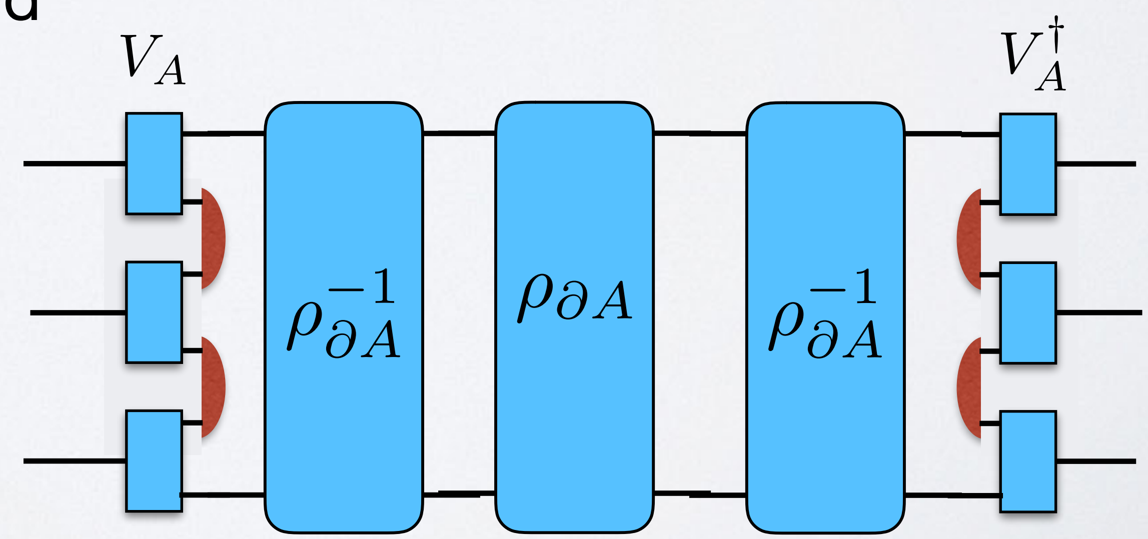

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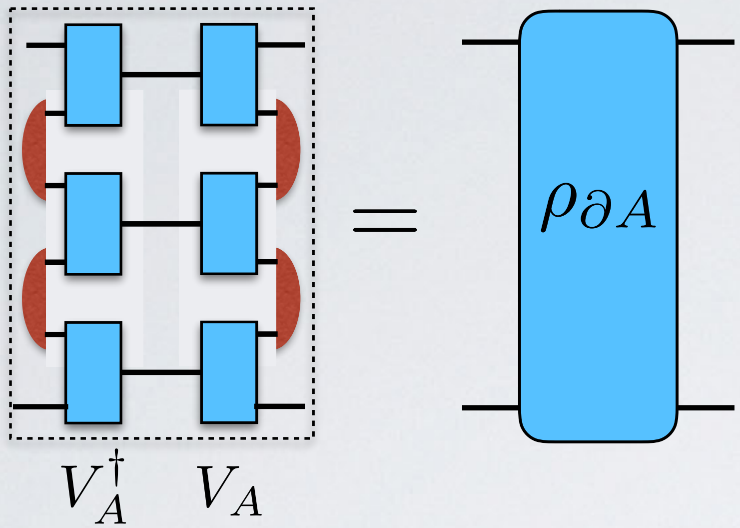

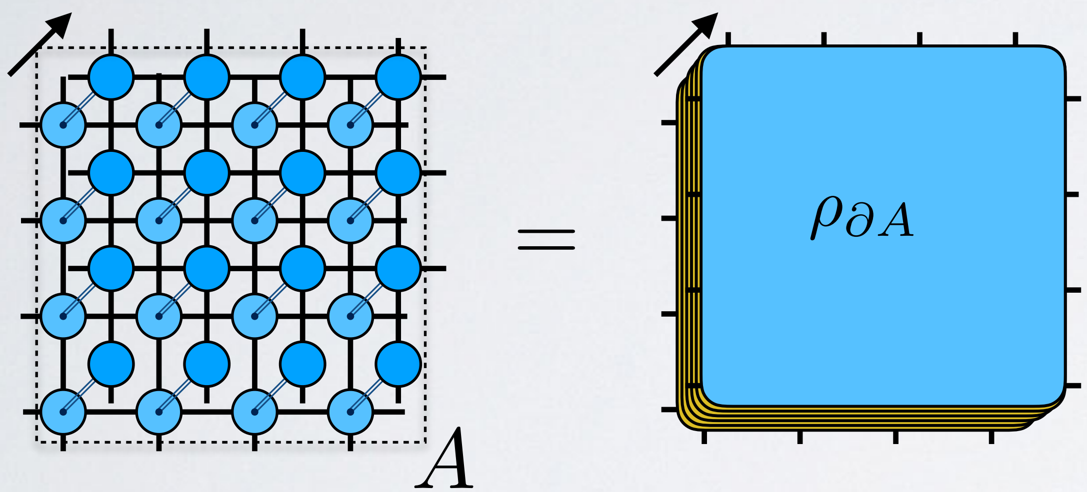

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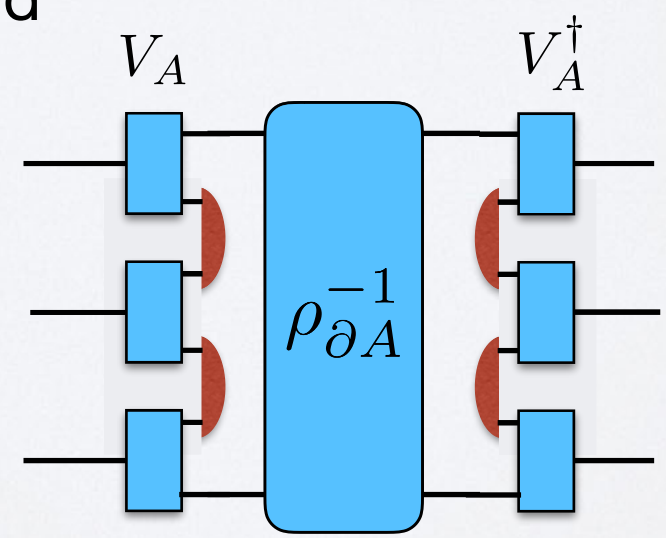

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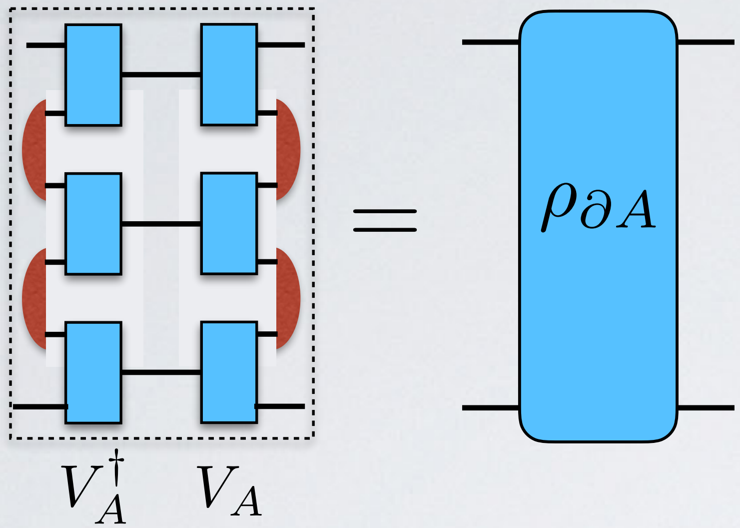

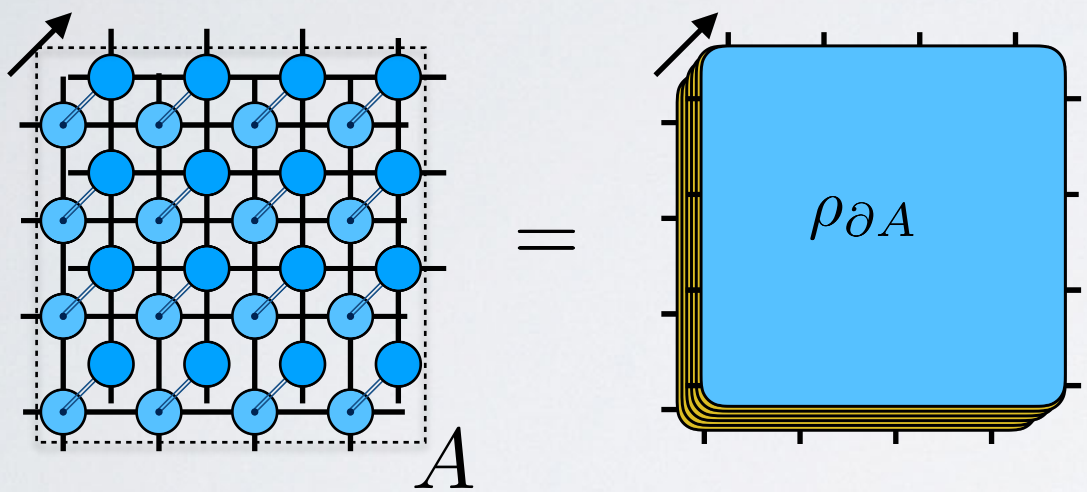

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Observe: P_A projects onto the ground state subspace of $H_A := \sum_{e \in E_\Lambda} h_e$

OUTLINE

If the boundary state of an (injective) PEPS is Gibbs (or sufficiently local), then its parent Hamiltonian is gapped.

1) PEPS basics

2) Spectral gap for frustration free Hamiltonians

PROOF OF THE GAP

- The *spectral gap* $\lambda(\Lambda)$ of a frustration-free Hamiltonian H_Λ is the smallest non-zero eigenvalue of H_Λ. If $\inf_\Lambda \lambda(\Lambda) > 0$ then we say that H is gapped.

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- The 'Martingale' gap theorem:

Theorem: if for three adjacent rectangles $A, B, C \subset \Lambda$, and some constants $\alpha, c > 0$, the following holds

$$||P_{AB}P_{BC} - P_{ABC}|| \leq c\ell_B^{-\alpha}$$

Then H is gapped.

Nachtergaele, '92,
K., Brandao, '15,
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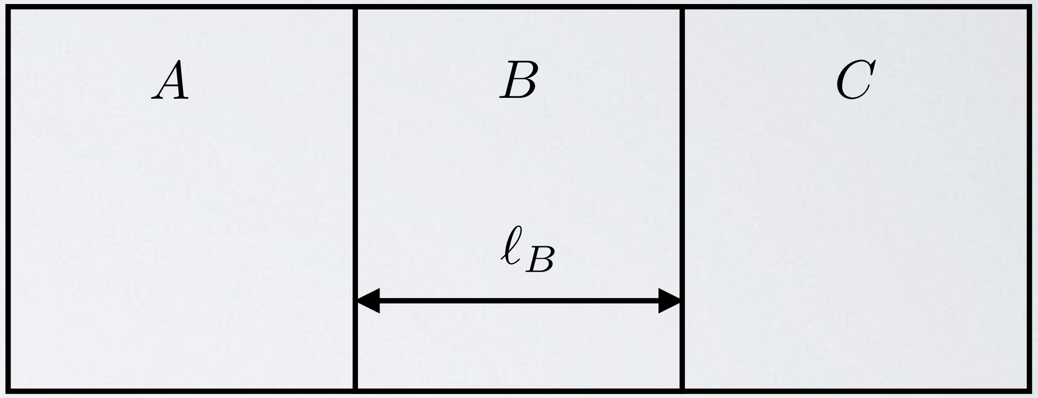

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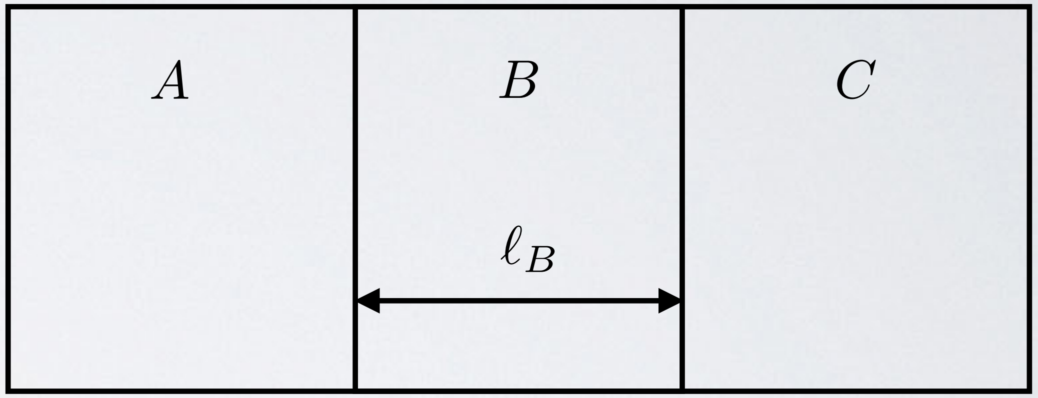

➡ Conversely, if H is uniformly gapped then

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➡ Quantum extension of the theorems relating spacial mixing to convergence of Glauber dynamics for ferromagnetic Ising model.

THE MAIN THEOREM

3) The main theorem

If the boundary state of an (injective) PEPS is Gibbs (or sufficiently local), then its parent Hamiltonian is gapped.

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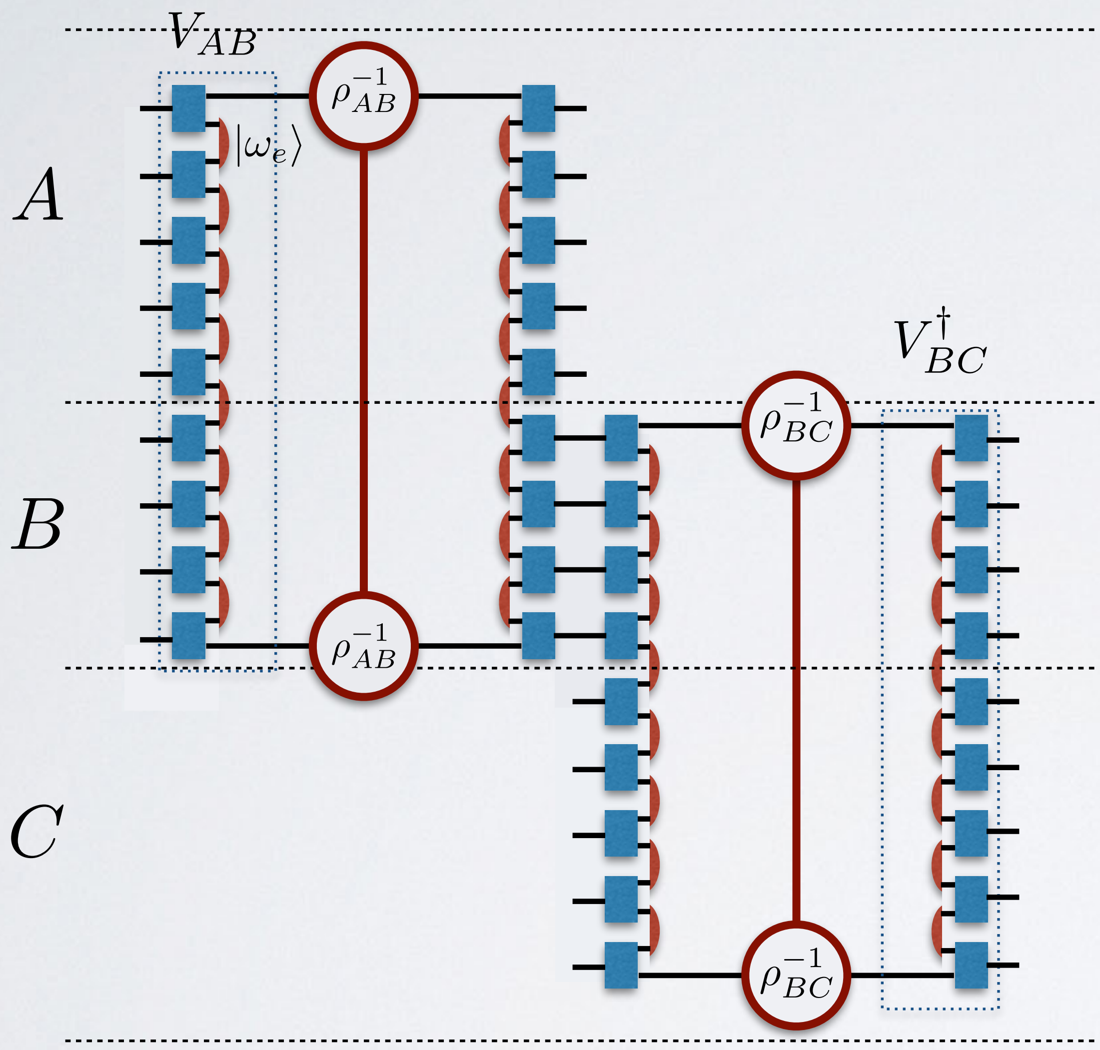
2) Spectral gap for frustration free Hamiltonians

THE MAIN THEOREM

- Sketch of the one dimensional case $P_{AB}P_{BC} \approx P_{ABC}$



$P_{AB}P_{BC}$

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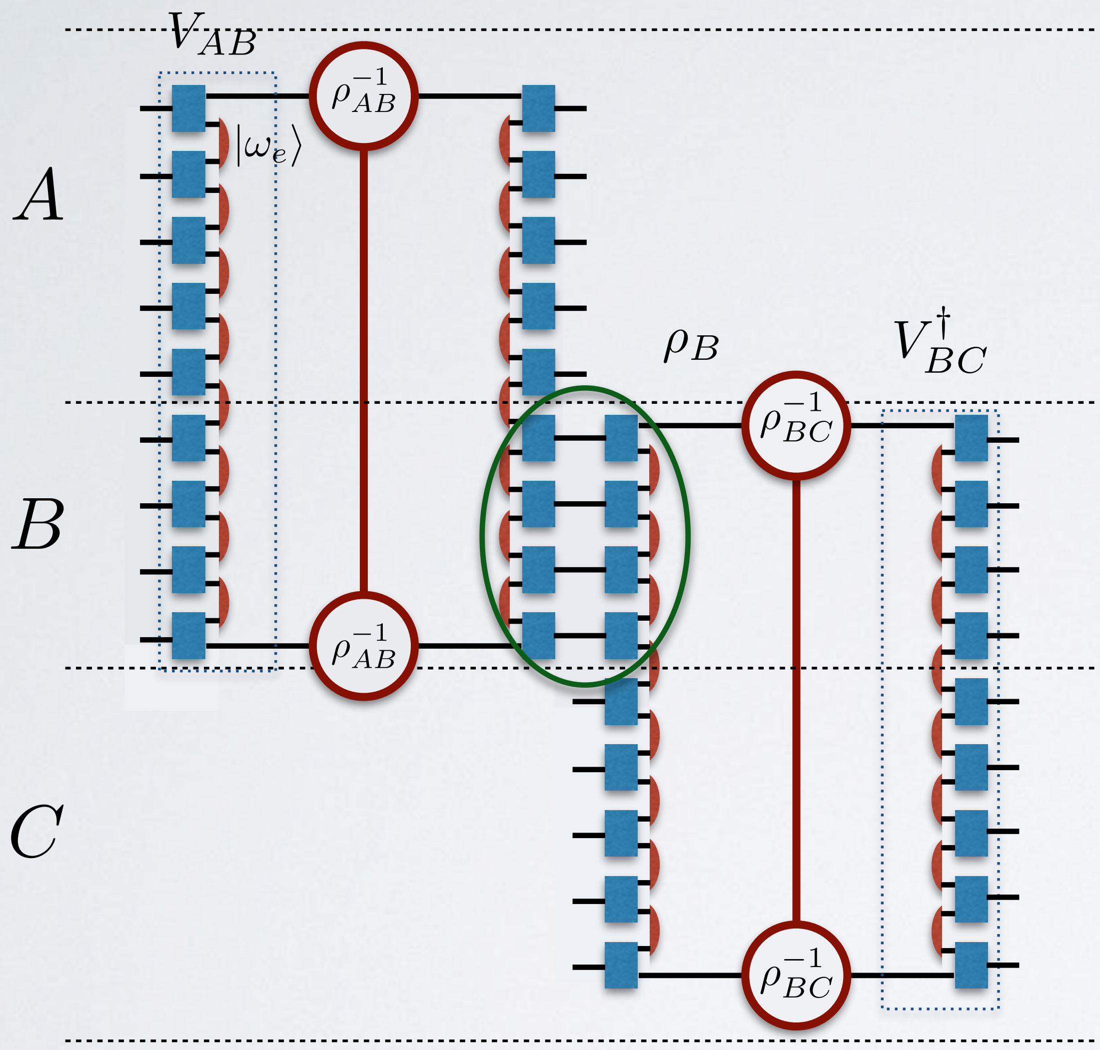

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THE MAIN THEOREM

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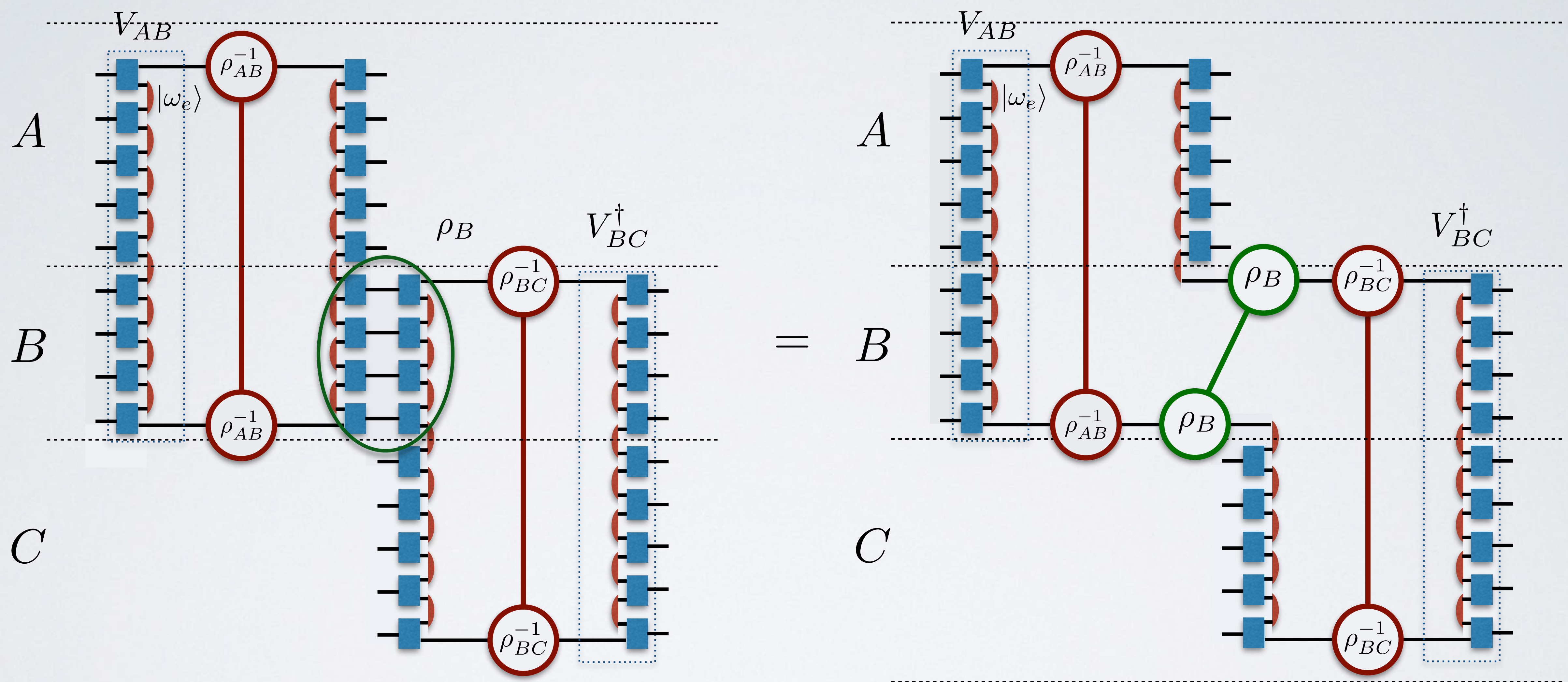

$P_{AB}P_{BC}$

$V_{AB}\rho_{AB}^{-1}V_A^{\dagger}\rho_B V_C \rho_{BC}^{-1}V_{BC}^{\dagger}$

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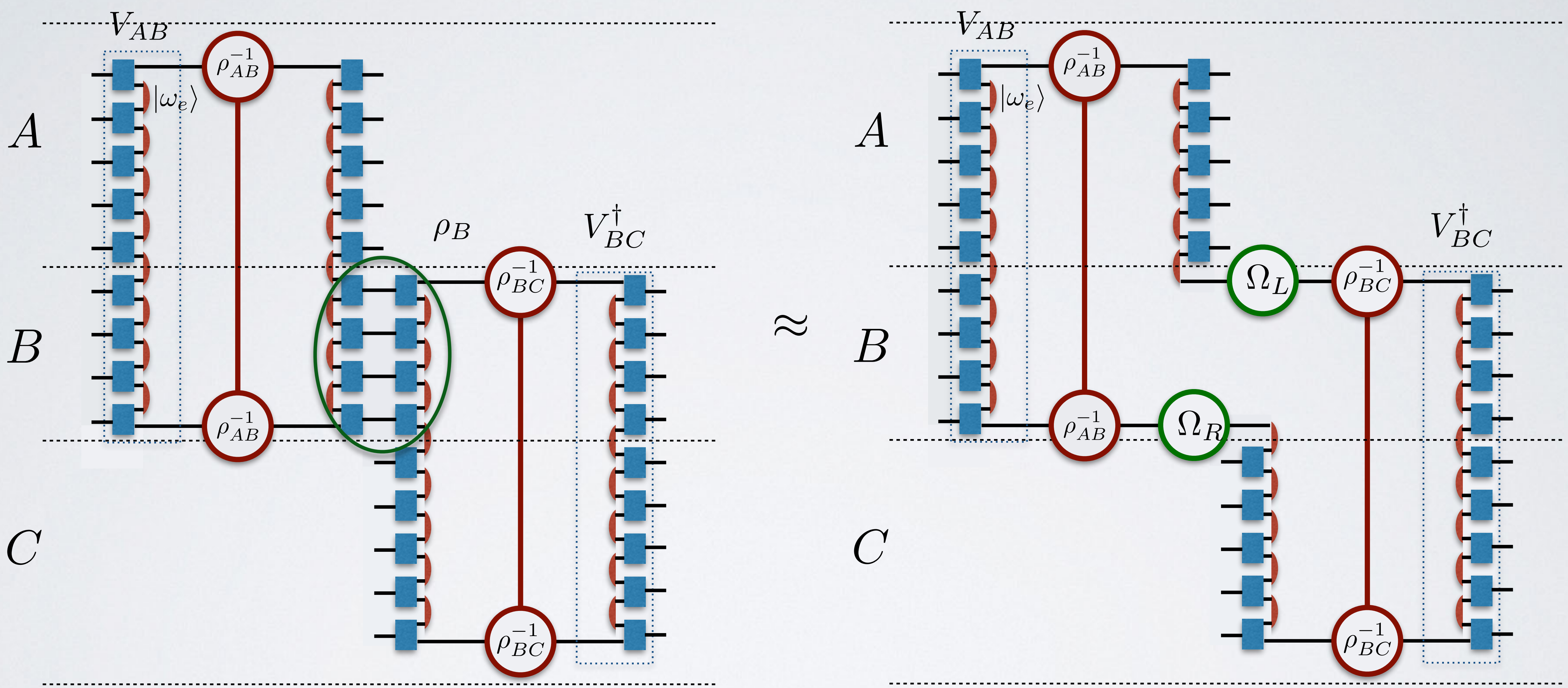

$P_{AB}P_{BC}$

$V_{AB}\rho_{AB}^{-1}V_A^\dagger\tilde{\rho}_B V_C \rho_{BC}^{-1} V_{BC}^\dagger$

$$\tilde{\rho}_B = \Omega_L \Omega_R$$

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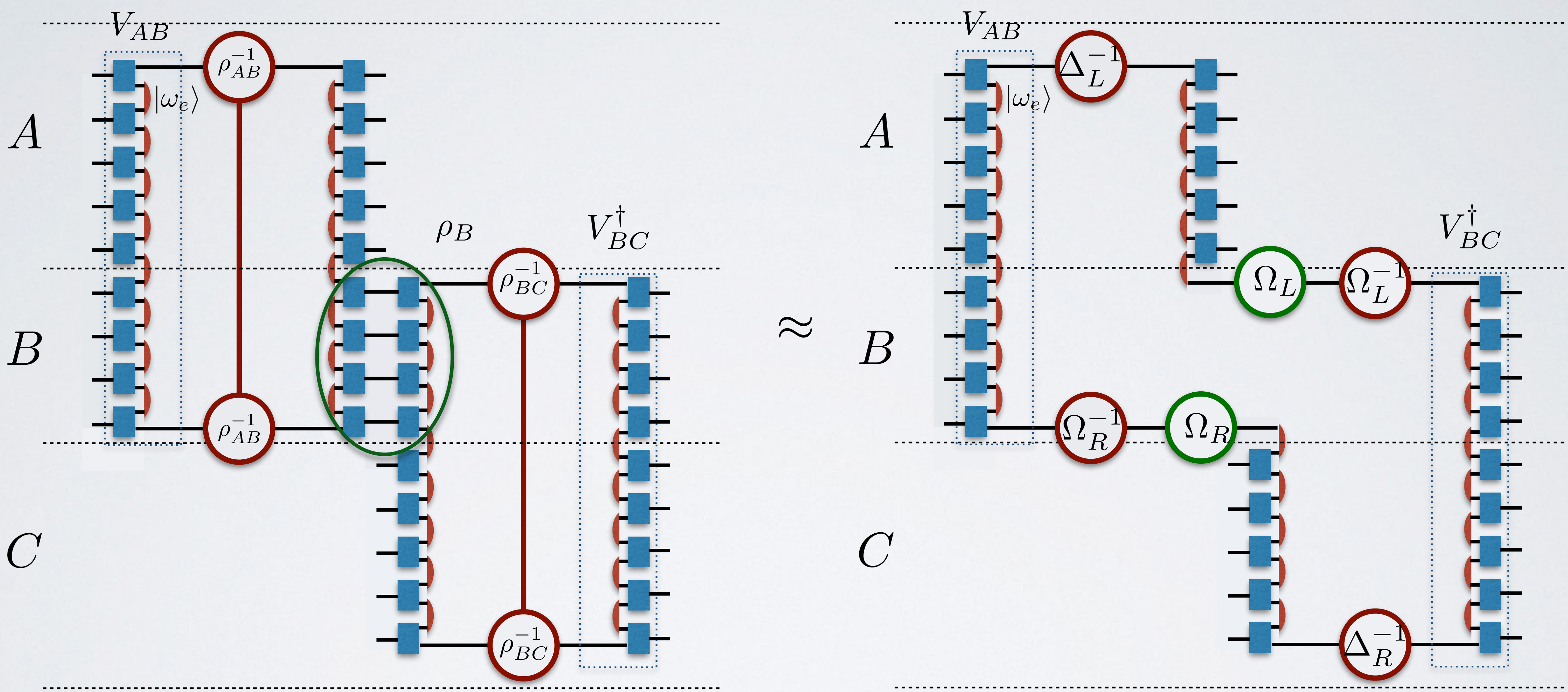

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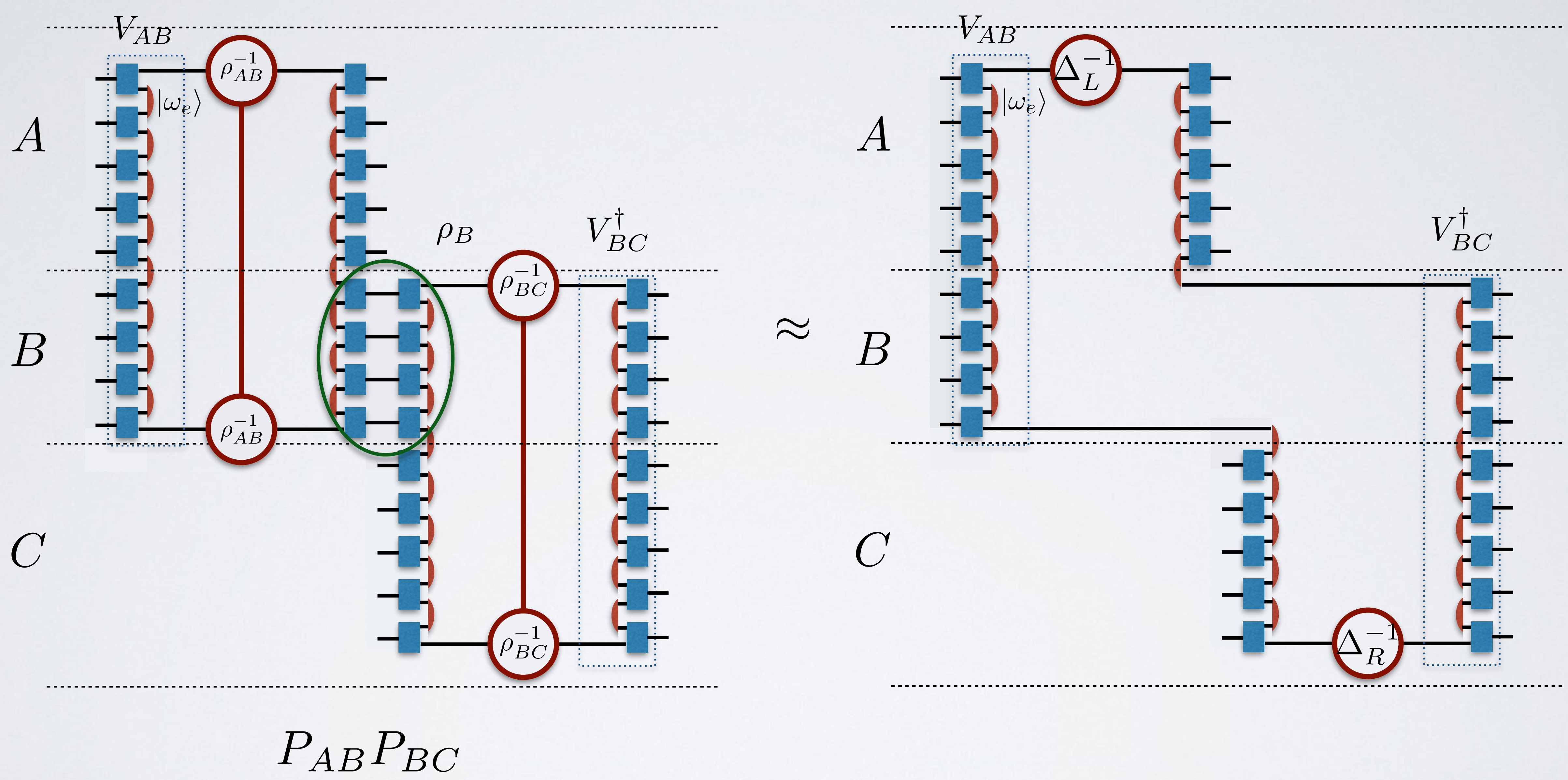

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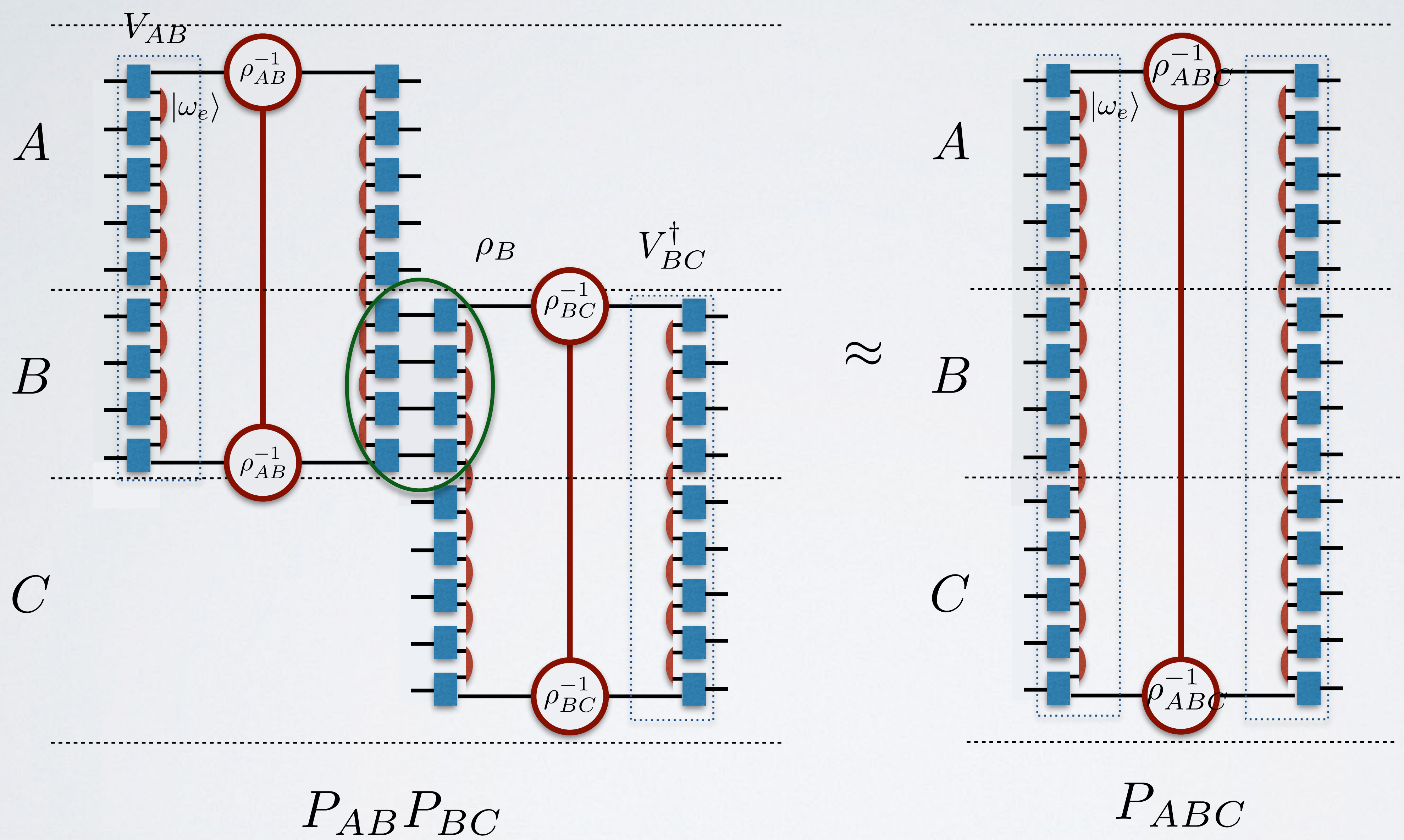

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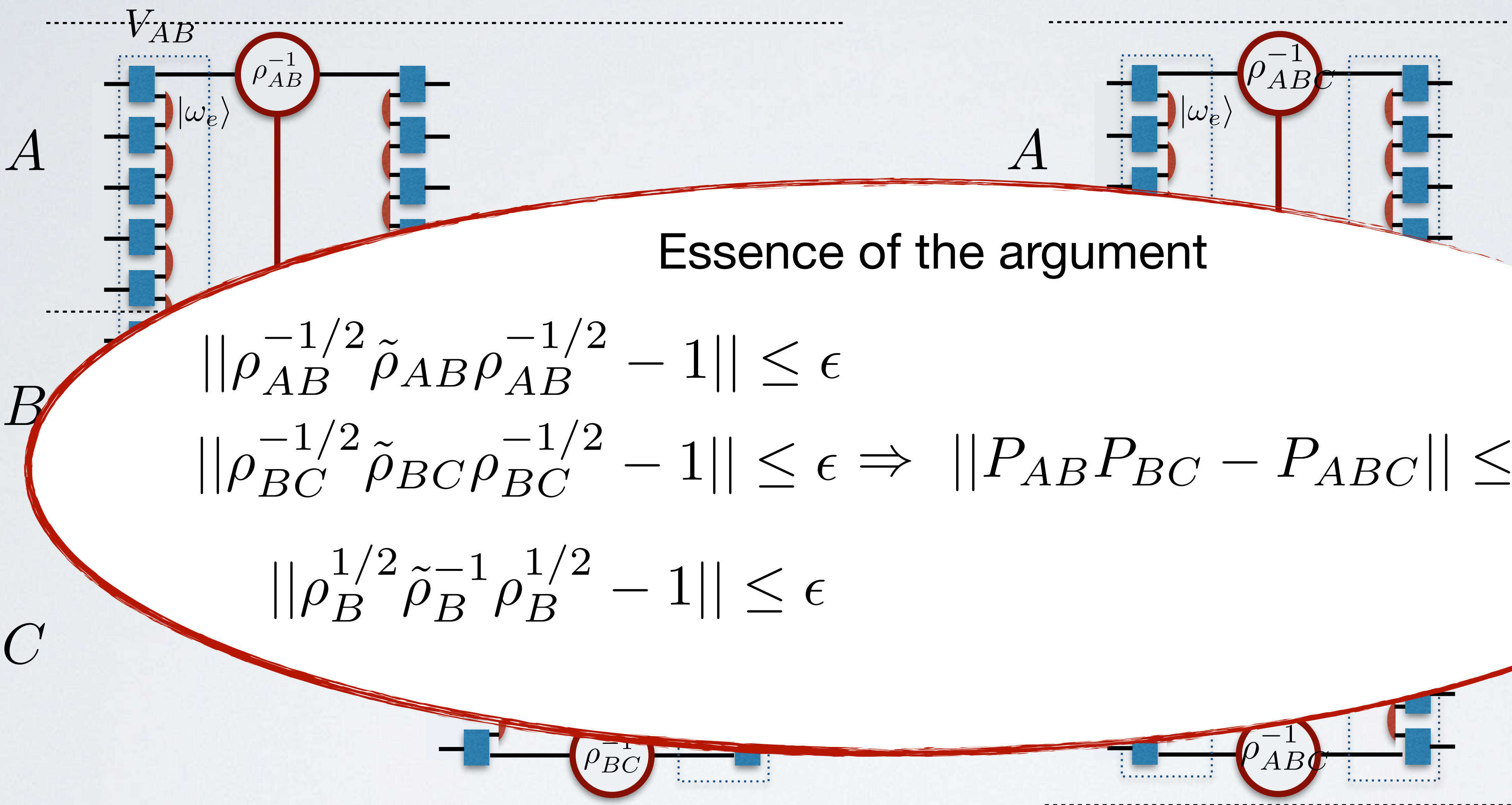

Essence of the argument

$$||\rho_{AB}^{-1/2}\tilde{\rho}_{AB}\rho_{AB}^{-1/2} - 1|| \leq \epsilon$$

$$||\rho_{BC}^{-1/2}\tilde{\rho}_{BC}\rho_{BC}^{-1/2} - 1|| \leq \epsilon \Rightarrow \; ||P_{AB}P_{BC} - P_{ABC}|| \leq 8\epsilon$$

$$||\rho_{B}^{1/2}\tilde{\rho}_{B}^{-1}\rho_{B}^{1/2} - 1|| \leq \epsilon$$

$P_{AB}P_{BC}$ $\qquad$ P_{ABC}

$$\tilde{\rho}_{AB}^{-1} = \Delta_L^{-1}\Omega_R^{-1} \quad \tilde{\rho}_{BC}^{-1} = \Omega_L^{-1}\Delta_R^{-1}$$

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THE MAIN THEOREM

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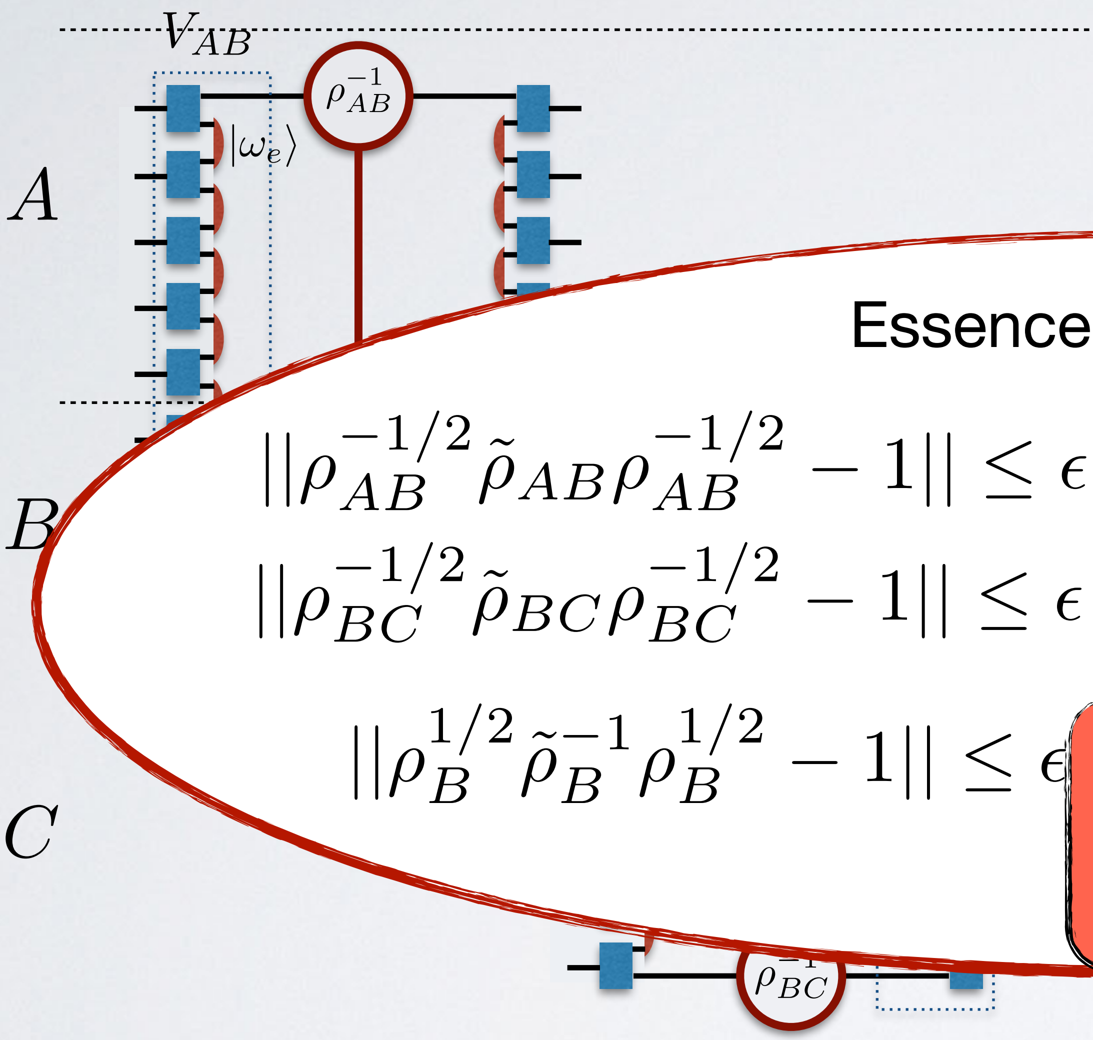

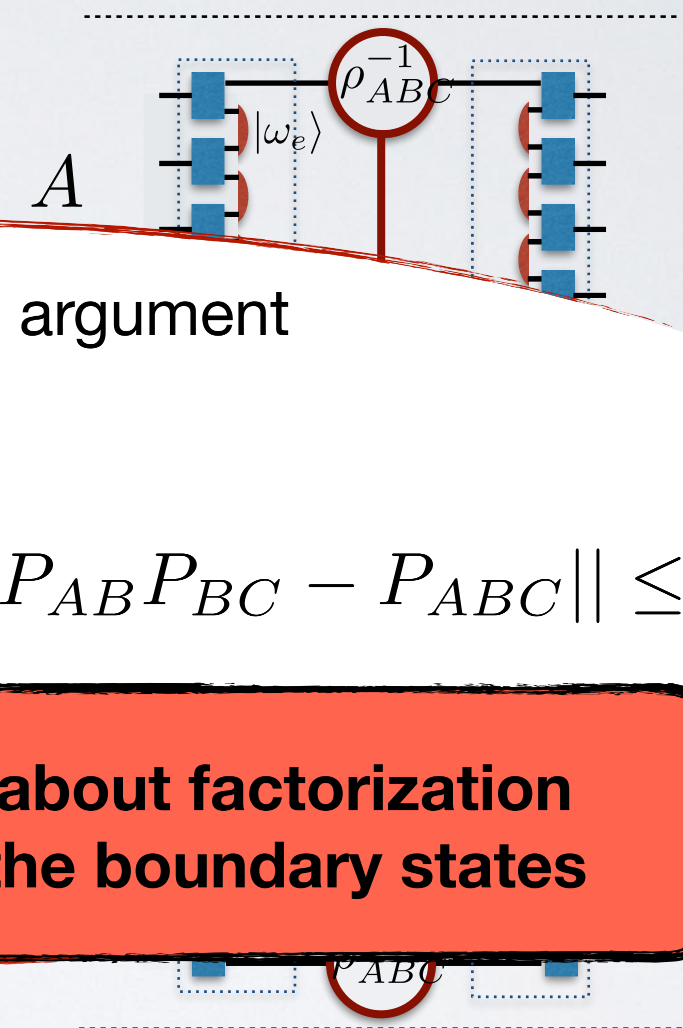

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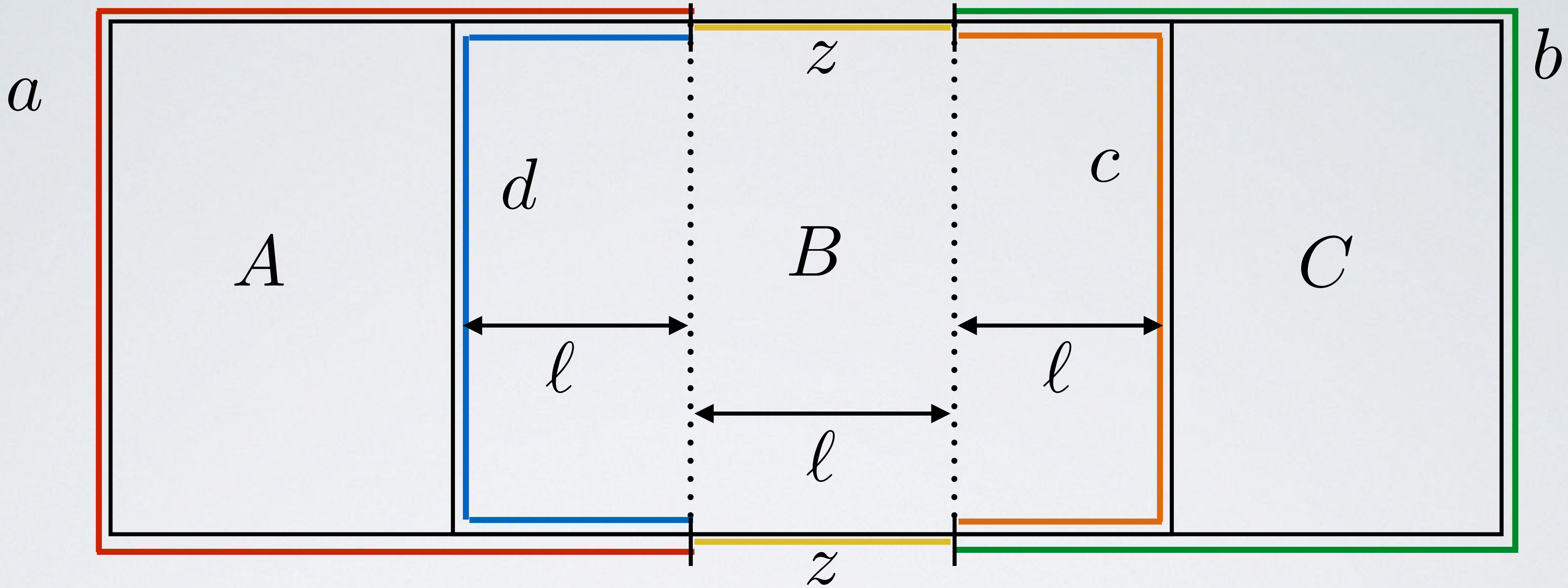
All about factorization of the boundary states

$P_{AB}P_{BC} \qquad P_{ABC}$

$$\tilde{\rho}_{AB}^{-1} = \Delta_L^{-1}\Omega_R^{-1} \qquad \tilde{\rho}_{BC}^{-1} = \Omega_L^{-1}\Delta_R^{-1}$$

$$\tilde{\rho}_B = \Omega_L\Omega_R$$

TWO DIMENSIONS



$$\tilde{\rho}_{AB}^{-1} = \Delta_{az}^{-1}\Omega_{zc}^{-1} \qquad \tilde{\rho}_{BC}^{-1} = \Omega_{dz}^{-1}\Delta_{zb}^{-1} \qquad \tilde{\rho}_{B} = \Omega_{zc}\Omega_{dz}$$

$$||\tilde{\rho}_X^{-1/2}\rho_X\tilde{\rho}_X^{-1/2} - 1|| \leq 1$$

$$X \in \{ABC, AB, BC, B\}$$

Factorization property

OUTLINE

4) The boundary states

3) The main theorem

If the boundary state of an (injective) PEPS is Gibbs (or sufficiently local), then its parent Hamiltonian is gapped.

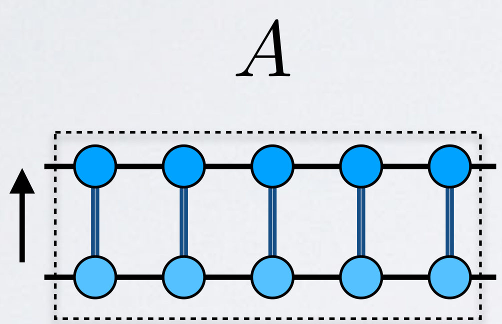
1) PEPS basics

2) Spectral gap for frustration free Hamiltonians

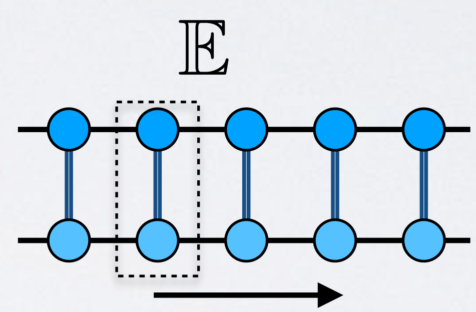
THE BOUNDARY STATES

When is the factorization property satisfied? → One dimensional (MPS)

- In 1D, the boundary state is the Choi–Jamiołkowski state of (a power of) the transfer operator:



$= \rho_{\partial A}$



$= \mathbb{E}^5$

- For injective MPS, E can be made to be a *primitive* completely positive trace preserving map, and

→ $\mathbb{E}^n \to \mathbb{E}^\infty = \mathrm{tr}[\cdot]\sigma$ Exponentially in *n*

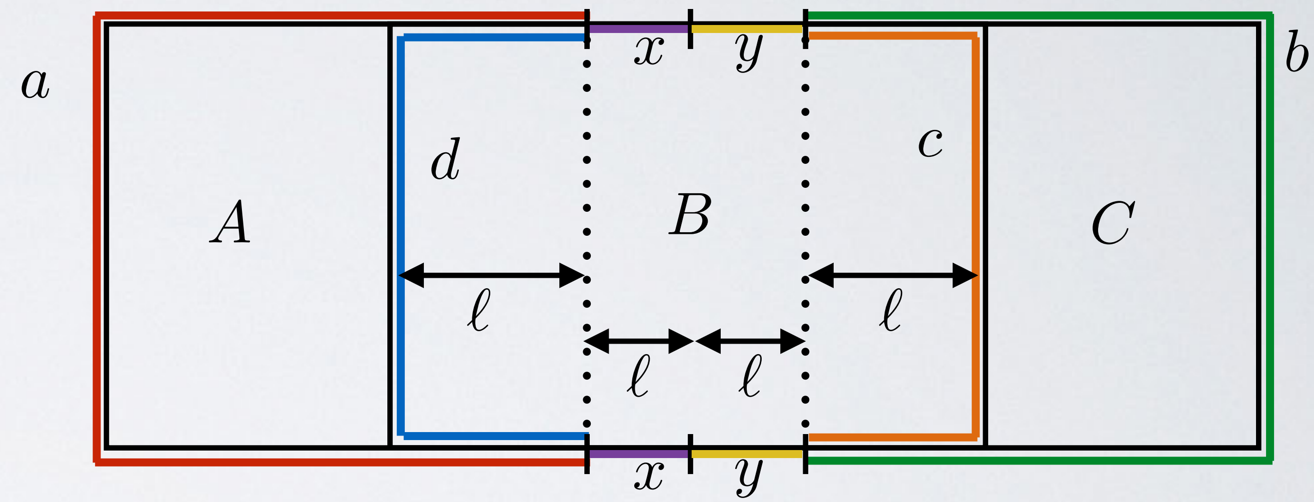
→ $||\rho_{\partial[1,n]} - 1 \otimes \sigma|| \to 0$ Exponentially in *n*

THE BOUNDARY STATES

When is the factorization property satisfied? ➡ Gibbs states at the boundary

- Suppose the boundary state is a local Gibbs state:

$$\rho_{\partial ABC} = e^{2H_{ABC}} \qquad H_{ABC} = \sum_{Z \subset axyb} h_Z$$



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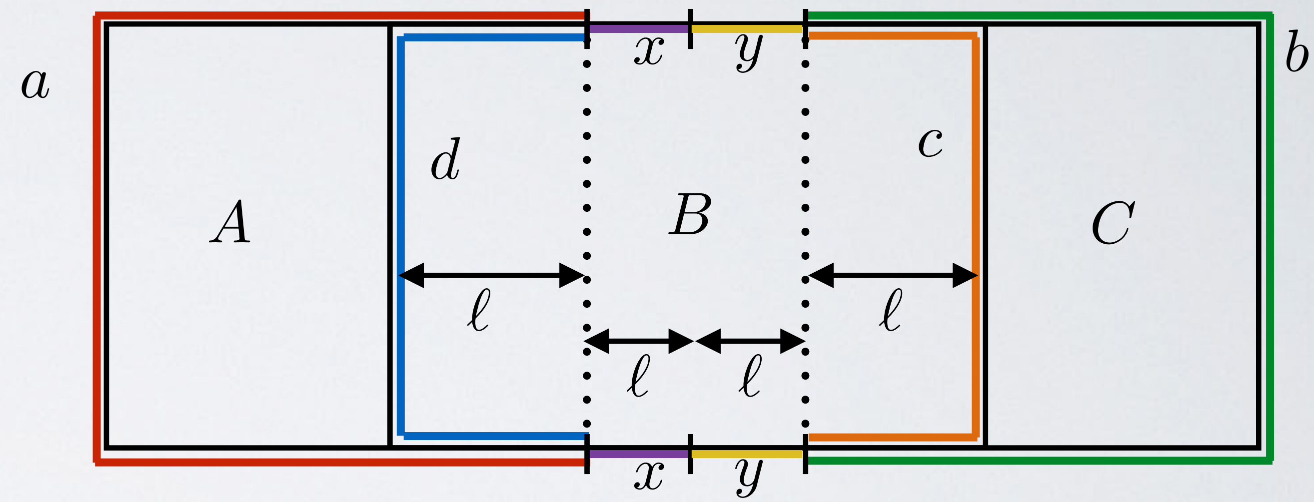

- Construct the following operators

$$\rho_{\partial ABC}^{1/2} \tilde{\rho}_{\partial ABC}^{-1} \rho_{\partial ABC}^{1/2} := e^{H_{axyb}} \underbrace{e^{-H_{axy}} e^{H_y} e^{-H_{ax}}}_{\Omega_L^{-1}} \underbrace{e^{-H_{yb}} e^{H_x} e^{-H_{xyb}}}_{\Omega_R^{-1}} e^{H_{axyb}}$$

$$= \left(e^{H_{axyb}} e^{-H_{axy}} e^{H_y} e^{-H_{yb}}\right) \left(e^{-H_{ax}} e^{H_x} e^{-H_{xyb}} e^{H_{axyb}}\right)$$

Each bit separately close to identity!

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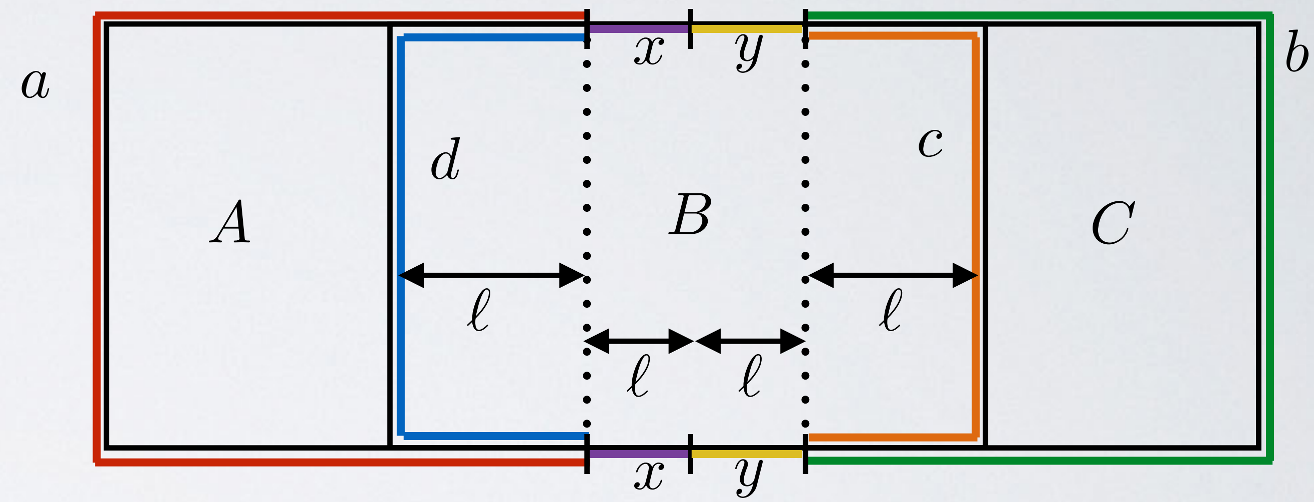


- $e^{H_{axyb}} e^{-H_{axy}} e^{H_y} e^{-H_{yb}} \approx 1$

Is based on Araki's proof of analycity of imaginary time evolution in 1D!

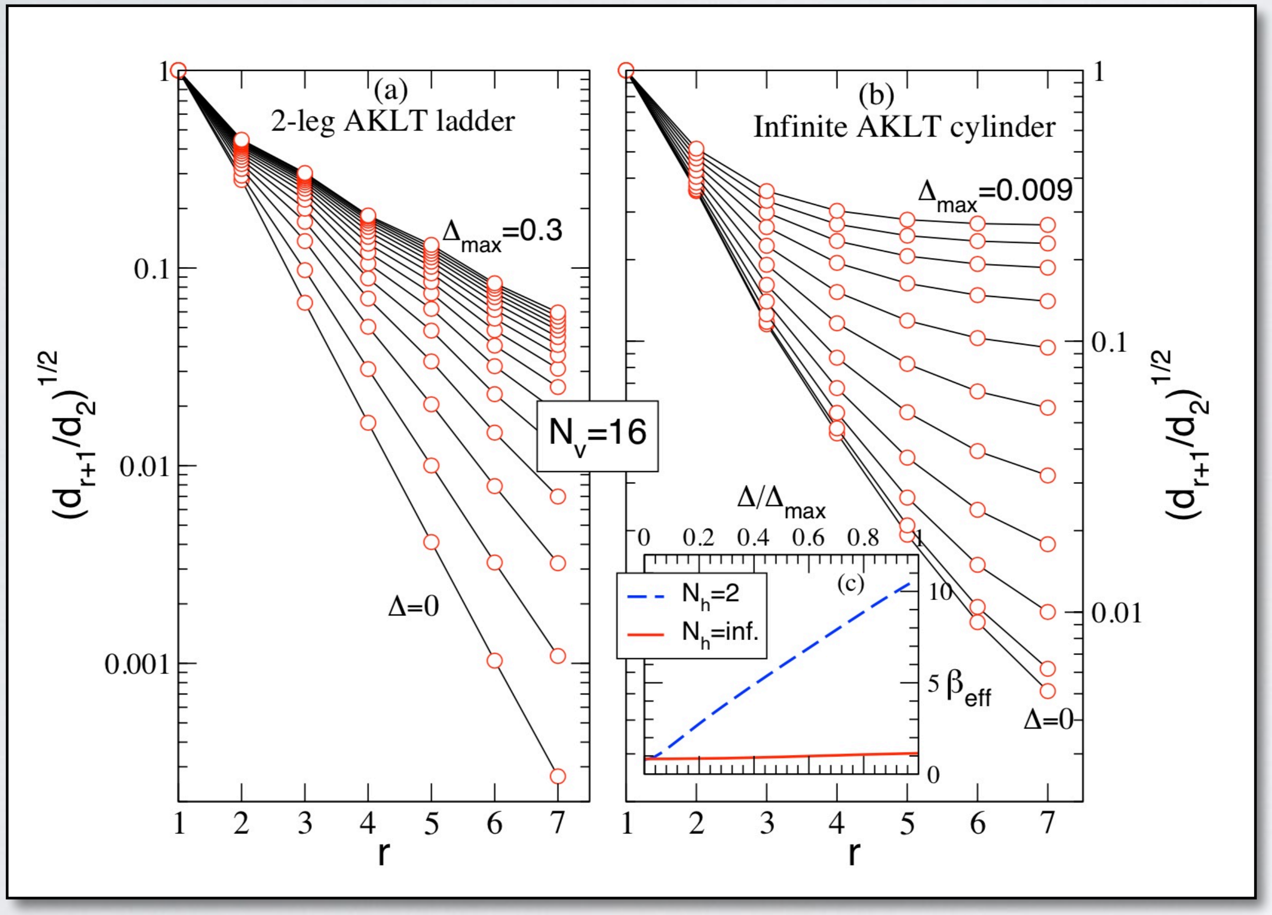
➡ Reminiscent of the conditional mutual information $I_\rho(A:C|B) = \mathrm{tr}[\rho(-\log \rho_{ABC} + \log \rho_{AB} - \log \rho_B + \log \rho_{BC})]$

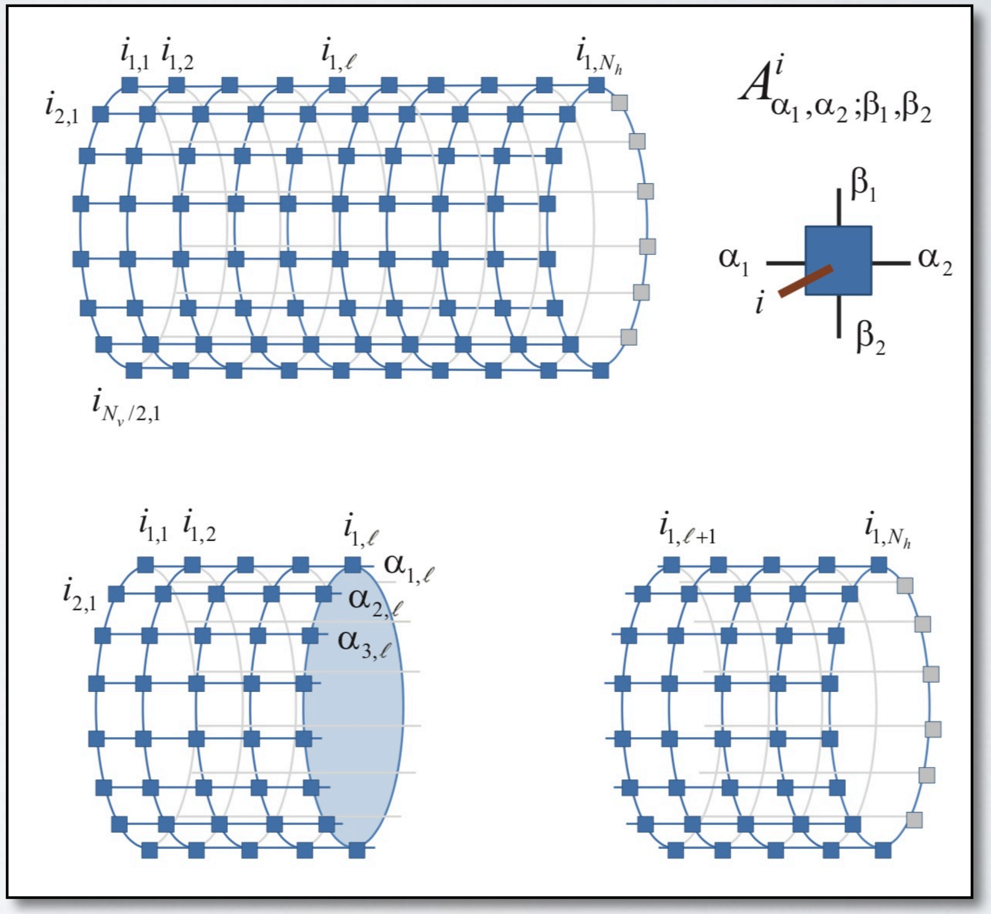
➡ We don't know to what extent the two are equivalent?

THE BOUNDARY STATES

Why should we believe the boundary state is Gibbs?

- Numerics on the AKLT model





Modelling Hamiltonian:

$$H_\partial = \sum_{r,k} d_r \mathbf{S}_k \cdot \mathbf{S}_{k+r} + R$$

- Unfortunately, exponential tails break our argument (since Araki's analycity techniques to not extend to exponential tails!)

OUTLOOK

- Can a converse be shown: that a gapped bulk implies quasi-factorisation of the boundary sates?
- Can quasi-factorisation be shown for specific models?
  - ➡ AKLT on the hexagonal lattice?
- Normalization or temperature of the boundary state?
- Relation to other static properties, such as LTQO?
- Can similar bulk-boundary correspondences be shown beyond the setting of PEPS?

THANK YOU!